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Dual sets and dual correspondences and their application to equilibrium theory

Research memorandum

TILBURG INSTITUTE OF ECONOMICS
DEPARTMENT OF ECONOMETRICS

DUAL SETS AND DUAL CORRESPONDENCES, AND THEIR APPLICATION TO EQUILIBRIUM THEORY.

by

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Introduction.

This paper consists of two parts. In part I the mathematical concept of duality is analyzed and in part II duality is applied to economics.
In the first part two types of dual sets are introduced, upper and lower dual sets. Different properties are given and their relation to dual cones is analyzed. The concept of dual summation is defined and it is shown that the dual of a sum of sets is equal to the dual sum of their duals. Intersection properties of sets and their duals are considered. Dual correspondences are defined as correspondences having the duals of the image of the original correspondence as their image and it is shown that, given certain assumptions, the dual of a closed correspondence is lower hemi continuous and vice versa.
In the second part an economy is defined and the dual representation of this economy is derived. The original representation being (mainly) in terms of commodity vectors, the dual representation is in terms of price vectors. Upper dual sets are applied to preferences, lower dual sets to production. For the original representation and for the dual representation a set of assumptions is given, the latter set being implied by the first. For both economics an equilibrium is defined, a dual equilibrium consisting of a price vector only. It is shown that both equilibria are equivalent. The existence of a dual equilibrium is proved.
This paper is an extension of [13]. The treatment of duality is more systematic and the theorems on intersection properties and dual summation are extended.
Dual correspondences are new. The economic model is more general, since the assumptions are weakened. The existence proof is different and based on the properties of dual correspondences.

*) I thank Pieter Ruys for his comments and his helpful suggestions.

Duality was applied to utility functions by Roy [7], and applied to preferences in [5] and [12]. An extensive study with respect to production functions can be found in [10].
The mathematical concept of duality can be found in [4] and [11]. Duality is applied in the theory of public goods in [8] and to the theory of adjoint correspondences in [9] (see remark section 11).

PART I

1. Some definitions.

A set $K \subset R^n$ is called a cone (with respect to the origin), if $x \in K \Rightarrow \lambda\ x \in K$ for all $\lambda \geq 0$. It is called an aureoled set, if $x \in K \Rightarrow \lambda\ x \in K$ for all $\lambda \geq 1$ and it is called a star shaped set if $x \in K \Rightarrow \lambda\ x \in K$ for all $0 \leq \lambda \leq 1$.
We define three closure operations, which associate the smallest set of each type to any set $C \subset R^n$:

$$\text{Cone } C = \{x \in R^n \mid \exists \lambda \geq 0, \exists y \in C: x=\lambda y\}$$

$$\text{Au } C = \{x \in R^n \mid \exists \lambda \geq 1, \exists y \in C: x=\lambda y\}$$

$$\text{St } C = \{x \in R^n \mid \exists 0 \leq \lambda \leq 1, \exists y \in C: x=\lambda y\}.$$

Obviously, if C is convex, all three closures are also convex and C = Au C $\cap$ St C .We have Cone C = Au(St C) = St(Au C) = Au C $\cup$ St C. We also define the set Coneint C, i.e. the largest cone, which is contained in C:

$$\text{Coneint } C = \{x \in R^n \mid \forall \lambda \geq 0: \lambda\ x \in C\}.$$

The sets Cl Cone C, i.e. the smallest closed cone, containing K, and the set Cl Coneint K, the closure of the "interior cone" happen to be nearly related to asymptotic cones.
We first define: let $k \in R$ and $C_k = \{x \in C \mid |x| \geq k\}$.
Then the asymptotic cone Asc C = $\bigcap_k$ Cl Cone C_k.

Property 1.1.

a) If C is aureoled, then Asc C = Cl Cone C

b) if C is star shaped, the Asc C = Cl Coneint C.

Proof

a) Asc C $\subset$ Cl Cone C : since $\forall k: C_k \subset$ Cl Cone C
Asc C $\supset$ Cl Cone C: if $x \in$ Cone C, then there exists k, such that for any $k' \geq k: x \in$ Cl Cone $C_{k'}$, hence

Asc C $\supset$ Cone C and so also its closure.

b) Cl Coneint C $\subset$ Asc C: $\forall k$:Coneint C $\subset C_k$
Cl Coneint C $\supset$ Asc C:Let $x \notin$ Cl Coneint C, then there exists $y \notin$ Cl C, such that $y=\lambda x$, for some $\lambda > 1$.
So for some k, $k' \geq k \Rightarrow y \notin C'_k$, so $x \notin$ Asc C .

2. Hyperplanes.

Let R^n and R^{n*} be two "different" n-dimensional spaces, which are distinguished only for reasons of interpretation. R^n is called the "original" space or the "commodity" space and R^{n*} is the "dual" space or the "price" space.
On $R^n \times R^{n*}$ the scalar product $px = \sum_{k=1}^{n} p^k x^k$ is defined.
Now for $p \in R^{n*}$ and $\alpha \in R$ we define $(p \neq 0)$

$$H(p,\alpha) = \{x \in R^n / px=\alpha\}.$$

The n-1-dimensional hyperplane $H(p,\alpha)$ separates the half spaces $\{x/px \geq \alpha\}$ and $\{x/px \leq \alpha\}$. Similarly for $x \in R^n$ and $\alpha \in R$ $(p \neq 0)$

$$H(x,\alpha) = \{p \in R^{n*}/px=\alpha\}.$$

We also define for $p \in R^n$ and $p \neq 0$:

$$L(p) = \{x \in R^n / px=1\}.$$

and we have $L(p) = H(p,1) = H(\alpha p,\alpha)$ and $H(p,\alpha) = H(\frac{1}{\alpha}p,1) = L(\frac{1}{\alpha}p)$.
$L(x)$ is defined by interchanging x and p.
Given $H(p,\alpha)$ and a set $C \subset R^n$, there are four possibilities:

1) The hyperplane intersects the set in its interior:
$H(p,\alpha) \cap \text{Int } C \neq \emptyset$

2) The hyperplane supports C in some point $\bar{x}$: $\bar{x} \in H(p,\alpha) \cap C$ and $H(p,\alpha) \cap \text{Int } C = \emptyset$. Now $p\bar{x} = \max_{x \in C} px=\alpha$ or $p\bar{x} = \min_{x \in C} px=\alpha$

3) The hyperplane asymptotically supports C:$H(p,\alpha) \cap C = \emptyset$ and $\inf_{x \in C} px = \alpha$ or $\sup_{x \in C} px = \alpha$. Obviously C is unbounded.

4) Both sets do not intersect and $H(p,\alpha)$ is not an asymptotic support. In this case there exists some $\alpha' > \alpha$ or $\alpha'' < \alpha$ such that $H(p,\alpha') \cap C = \emptyset$ or $H(p,\alpha'') \cap C = \emptyset$.

3. Closed, convex, aureoled sets, not containing 0.

A certain type of set which will be frequently used in this paper is called a type A set.

Definition 3.1

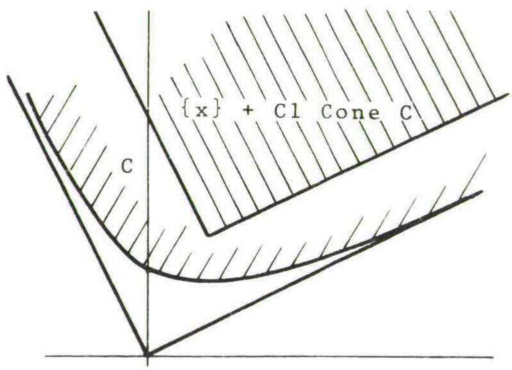
A set $C \subset R^n (C \subset R^{n*})$ will be called a type A set if $0 \notin C$ and C is closed, convex aureoled.

Type A sets have properties which are similar to properties of cones. For a closed cone, we have $K + K = K$ (see fig. 1).

Property 3.2

If C is a type A set, $C + \text{Cl Cone } C = C$.

Fig. 1



Proof

Obviously $C + \text{Cl Cone } C \supset C + \{0\} = C$. Conversely, we show that $C + \text{Cone } C \subset C$. Let $x \in C$ and $y \in \text{Cone } C$, where $\lambda y \in C$, for $\lambda \geq 1$. Now $\frac{\lambda}{1+\lambda}(x+y) = \left(\frac{\lambda}{1+\lambda}\right) x + \left(\frac{1}{1+\lambda}\right) \lambda y \in C$, since C is convex and $(x+y) \in C$, since $\frac{\lambda}{1+\lambda} < 1$ and C is aureoled. Since C is closed, we also have $C + \text{Cl Cone } C \subset C$.

For any $\alpha > 0$, we can define $\alpha C = \{x \mid \exists\, y \in C : x = \alpha y\}$. It is obvious that αC is also a type A set, and that $\text{Cl Cone } \alpha C = \text{Cl Cone } C$, and that $\alpha C \subset \beta C$ if $\alpha > \beta$.
It will be usefull to have a definition also for αC if $\alpha = 0$. First assume K is a cone. Since for any $\alpha > 0$, $\alpha K = K$, it seems

obvious to define $0K = K$. (See also [6], p 61). Further assume C is a type A set, $x \in C$ and $K = \text{Cl Cone } C$.
Now we have, for all $\alpha > 0$:

$$\{\alpha x\} + K \subset \alpha C + K = \alpha C \subset \alpha K = K$$

So it seems obvious to require $\{0x\} + K \subset 0C \subset K$, or $0C = K$.

Definition 3.3

If C is a cone, $0C = C$, if C is a type A set, $0C = \text{Cl Cone } C$.

If we have a finite member of type A sets, their sum is convex and aureoled. It is however not necessarily closed and it may contain zero. However if the sum of their closed cones is pointed, then the sum is a type A set.

Theorem 3.4

Let C_i $(i=1,2,...,n)$ be type A sets and $(\Sigma \text{ Cl Cone } C_i) \cap -(\Sigma \text{ Cl Cone } C_i) = \{0\}$, then ΣC_i is a type A set.

Proof

$0 \notin \Sigma C_i$: assume $0 = \Sigma x_i$ and $x_i \in C_i$. Now $x_i \neq 0$ and $x_1 = -\sum_2^n x_j$, hence $x_1 \in \Sigma \text{ Cl Cone } C_i$ and $x_1 = -\sum_2^n x_j \in - \Sigma \text{ Cl Cone } C_i$, which contradicts the assumption. Convex: $x = \Sigma x_i$, $y = \Sigma y_i$, for $x_i, y_i \in C_i$; now $\alpha x + (1-\alpha)y = \Sigma(\alpha x_i + (1-\alpha)y_i)$. Aureoled: $x = \Sigma x_i$, $\lambda x = \Sigma \lambda x_i$. Closed: in [2] is stated that a sum of closed convex sets is closed, if their asymptotic cones have the property of the theorem and we have shown that for type A sets the asymptotic cone is equal to the closed cone. (see [2], 1.9(9))

Property 3.5

If C_i are type A sets, then $\text{Cl Cone } \Sigma C_i = \Sigma \text{ Cl Cone } C_i$.

Proof

$C_i \subset \text{Cl Cone } C_i$, hence $\Sigma C_i \subset \Sigma \text{ Cl Cone } C_i$ and now $\text{Cl Cone } \Sigma C_i \subset \Sigma \text{ Cl Cone } C_i$

Let $x \in \Sigma$ Cone C_i, hence there exist x_i, such that $\Sigma x_i = x$ and $x_i \in$ Cone C_i. For som λ, $\lambda x_i \in C_i$, hence $\lambda x \in \Sigma C_i$, so Σ Cone $C_i \subset$ Cone ΣC_i and now Cl Σ Cone C_i = Σ Cl Cone $C_i \subset$ Cone ΣC_i.

4. Closed convex sets, containing 0.

An other type of set, frequently used in this paper and having properties similar to type A sets, will be called type S sets (since they are star shaped).

Definition 4.1

A set $Y \subset R^n$ will be called a type S set, if $0 \in Y$ and Y is closed and convex (see fig. 2).
Properties, analogous to the ones given in the previous section hold for these sets: Y+ Coneint Y = Y; a sum of type S sets is also a type S set, if the sum of their asymptotic cones is pointed and we may define 0 Y = Coneint Y.
Note that Coneint Y is closed for type S sets and that Coneint Y = $\emptyset$ if Y is compact.

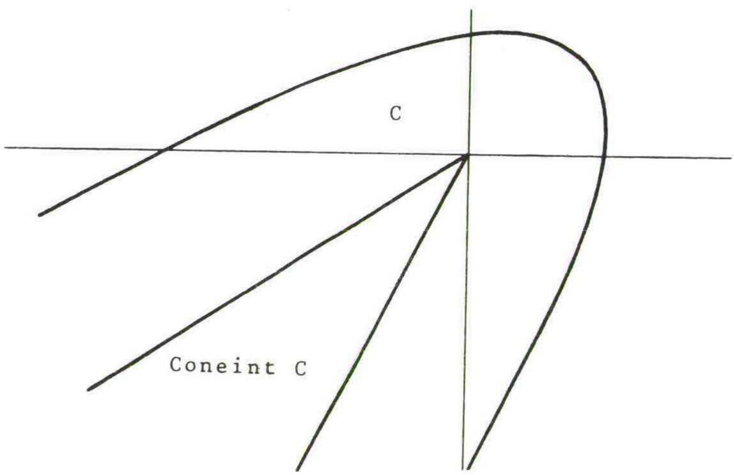


Fig. 2

5. Upper dual sets.

Let $C \subset R^n$ be any set. We define its upper dual set as $C_+^* \subset R^{n*}$, where

Definition 5.1

For $C \subset R^n$, $C_+^* = \{p \in R^n \mid \forall x \in C: px \geq 1\}$.

C_+^* contains all
$p \in R^{n*}$, such that the hyperplane $L(p)$ (see section 2) separates C and 0. This directly implies, that $C_+^* \neq \emptyset$ if and only if $\mathrm{Cl\ Conv}\ C \not\ni 0$. If a hyperplane $L(p)$ supports or asymptotically supports C, then p is a boundary point of C_+^*, if $L(p)$ contains an interior point of C, then p is not in C_+^* (see fig. 3)

The above definition gives C_+^* as a subset of R^{n*} for $C \subset R^n$.
If however $B \subset R^{n*}$, then B_+^* is in the original space:
$B_+^* = \{x \in R^r \mid \forall\ p \in B: px \geq 1\}$.
Hence $(C_+^*)_+^* = C_{++}^{**}$, the dual of the dual, is in the original space.

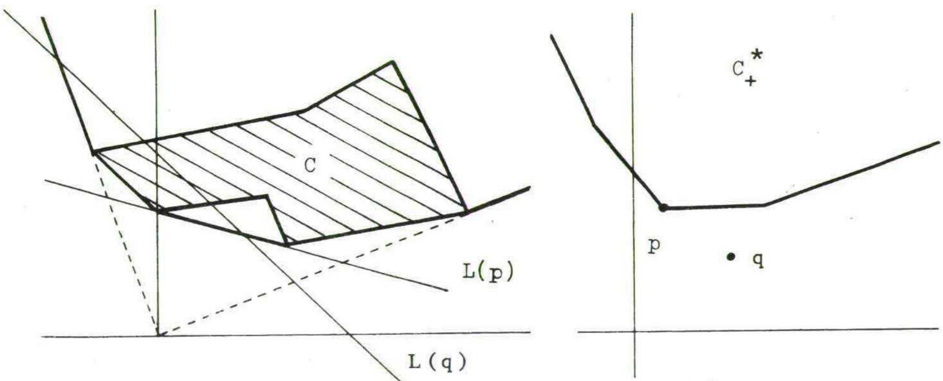


Fig. 3

Property 5.2

If $C \subset D$, then $C_+^* \supset D_+^*$.

Proof

Let $p \in D_+^*$, hence $\forall x \in D$: $px \geq 1$ and therefore also $\forall\ x \in C$: $px \geq 1$.

Property 5.3

For any $C \subset R^n$, C_+^* is a type A set.

Proof

$0 \notin C_+^*$: obvious. Convex: if for all $x \in C, px \geq 1$ and $qx \geq 1$, then also $\alpha px+(1-\alpha)qx \geq 1$, for $\alpha \in [0,1]$.
Aureoled: if $\lambda \geq 1$, then $\forall x \in C$: $px \geq 1 \Rightarrow \forall\ x \in C:\lambda px \geq 1$.
Closed: assume $p \in \mathrm{Cl}\ C_+^*$ and $p \notin C_+^*$. Now there exists $x_o \in C$, such that $px_o < 1$, but then, for ε sufficiently small, $q \in B_\varepsilon(p) \Rightarrow qx_o < 1$, which is a contradiction.

From this property it directly follows, that C_{++}^{**} is also a type A set We have:

Property 5.4

For any $C \subset R^n$, $C \subset C_{++}^{**}$. If C is a type A set, then $C = C_{++}^{**}$.

Proof

$C \subset C_{++}^{**}$: Let $x_o \in C$, then by definition, $\forall p \in C_+^*: px_o \geq 1$. Hence $x_o \in C_{++}^{**} = \{x/\forall p \in C_+^*: px \geq 1\}$. $C \supset C_{++}^{**}$: Let $x_o \notin C$, and C a type A set. For $T = \{y=\alpha x_o \mid \alpha \in [0,1]\}$, $T \cap C = \emptyset$, since C is aureoled. As T is compact convex and C is closed and convex, there exists a hyperplane L(p), strictly separating T and C. Now $p \in C_+^*$ and since $px_o < 1, X_o \in C_{++}^{**}$.

From this property it follows, that $C_+^* = (\mathrm{Cl}\ C)_+^* = (\mathrm{Conv}\ C)_+^* = (\mathrm{Au}\ C)_+^*$ and if Cl C = Cl Int C, also $C_+^* = (\mathrm{Int}\ C)_+^*$. By applying 5.2, it also follows that $C_{++}^{**} = \mathrm{Cl\ Au\ Conv}\ C$.

Property 5.5

If C_i $(i \in I)$ is a (possibly infinite) family of sets, than

a) $(\bigcup_I C_i)_+^* = \bigcap_I C_{i+}^*$

b) if C_i are type A sets, $(\bigcap_I C_i)_+^* = \mathrm{Cl\ Conv} \bigcup_I C_{i+}^*$.

Proof

a) From 5.2 it follows, that $(\cup C_i)_+^* \subset C_i^*$, for all i, hence $(\cup C_i)_+^* \subset \cap C_{i+}^*$. Conversely , let $p \in \cap C_{i+}^*$, hence for all i, $x \in C_i \Rightarrow px \geq 1$, and therefore, $x \in \cup C_i \Rightarrow px \geq 1$, so $p \in (\cup C_i)_+^*$

b) By substituting C_{i+}^{*} for C_i in a), we get $(\cup C_{i+}^{*})_{+}^{*} =$ $\cap C_{i++}^{**} =$ $\cap C_i$, since all C_i are type A. By taking duals on both sides: $(\cap C_i)_{+}^{*} = (\cup C_{i+}^{*})_{++}^{**}$ = Cl Conv $\cup C_{i+}^{*}$, the union of aureoled sets being aureoled.

Note that it is not excluded, that $\cup C_i$ or $\cup C_{i+}^{*}$ contains 0. In this case its dual, and therefore the intersection, must be empty.

6. Lower dual sets.

With respect to lower dual sets, type S sets, as defined in section 4, play the same role as type A sets play with respect to upper dual sets. The difference between upper dual sets and lower dual sets is, that in the definition $\geq$ is replaced by $\leq$.

Definition 6.1

For any non empty set $Y \subset R^n$, $Y_{-}^{*} = \{p \in R^{n*} | \forall x \in Y: px \leq 1\}$.

Now obviously $Y_{-}^{*} \neq \emptyset$ since $0 \in Y_{-}^{*}$ for any $Y \subset R^n$.
Apart from 0, Y_{-}^{*} contains all p, such that the hyperplane L(p) has 0 and Y on one side. L(p) should not intersect Y in its relative interior and if it supports or asymptotically supports Y, $p \in$ Bnd Y_{-}^{*}.
All properties are similar to ones in section 5, as are their proofs

Property 6.2

$X \subset Y \Rightarrow X_{-}^{*} \supset Y_{-}^{*}$

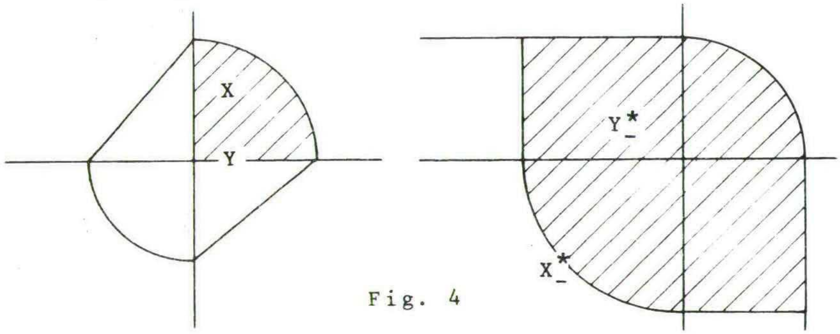


Fig. 4

Property 6.3

For any $Y \subset R^n$, Y_-^* is a type S set.

Property 6.4

$Y \subset Y_{--}^{**}$; if Y is a type S set, $Y = Y_{--}^{**}$

This implies that $Y_{--}^{**} = \mathrm{Cl\ Conv}\ \{\{0\},Y\}$

Property 6.5

a) If Y_i is a family of sets I, then $(\bigcup_I Y_i)_-^* = \bigcap_I Y_{i-}^*$

b) If Y_i are also type S, then $(\bigcap_I Y_i)_-^* = \mathrm{Cl\ Conv} \bigcup_I Y_{i-}^*$

7. Dual Cones.

We distinguish upper dual cones and lower dual cones. Their difference is however hardly relevant. An upper dual cone C_+^o of a set C, contains (besides 0) all p such the hyperplane H(p,0) has C on its positive side. The lower dual cone contains p, such that C is on the negative side of H(p,0).

Definition 7.1

For $C \subset R^n$,

$$C_+^o = \{p \in R^{n*} |\ \forall\ x \in C: px \geq 0\}$$
$$C_-^o = \{p \in R^{n*} |\ \forall\ x \in C: px \leq 0\}$$

Obviously $C_+^o = -C_-^o$ and $0 \in C_+^o$ and $C_+^o = \emptyset$ if $0 \in \mathrm{Int\ Cl\ Conv}\ C$. Their properties are well known and similar to the ones for upper dual sets (section 5) and lower dual sets (section 6). Their proofs parallel those of section 5. We only give the properties for C_+^o, those for C_-^o following by applying $C_+^o = -C_-^o$.

Property 7.2

$C \subset D \Rightarrow C^{o}_{+} \supset C^{o}_{+}$

Property 7.3

For any $C \subset R^{n}$, C^{o}_{+} is a closed convex cone.

Property 7.4

$C \subset C^{oo}_{++}$; If C is a closed convex cone, $C = C^{oo}_{++}$

and this implies

$C^{oo}_{++} = (\mathrm{Cl}\ C)^{o}_{+} = (\mathrm{Cone}\ C)^{o}_{+}$ and for C convex, $\mathrm{Cl\ Cone}\ C = C^{oo}_{++}$
and

Property 7.5

a) If C_i is a family of sets

$$(\cup\ C_i)^{o}_{+} = \cap\ C^{o}_{i+}$$

and if all C_i are closed convex cones:

$$(\cap\ C_i)^{o}_{+} = \mathrm{Cl\ Conv} \cup C^{o}_{i+}$$

Proof

a) By 7.3, $(\cup\ C_i)^{o}_{+} \subset \cap\ C^{o}_{i+}$. Assume $p \in \cap\ C^{o}_{i+}$, then $\forall i: x \in C_i \Rightarrow px \geq 0$, so $x \in \cup\ C_i \Rightarrow px \geq 0$, hence $p \in (\cup\ C_i)^{o}_{+}$

b) By a, $(\cup\ C^{o}_{i+})^{o}_{+} = \cap\ C^{oo}_{i++} = \cap\ C_i$, by 7.4. So $(\cap\ C_i)^{o}_{+} = (\cup\ C^{o}_{i+})^{oo}_{++} = \mathrm{Cl\ Conv} \cup C^{o}_{i+}$ (the convex hull of a union of cones being a convex cone).

Finally we have

Property 7.6

If C is a closed convex cone, and if $C \cap -C = \{0\}$, then $\mathrm{Int}\ C^{o}_{+} \neq \emptyset$.

Proof

Assume Int $C_+^o = \emptyset$. Then there exists a subspace of dimension $m < n$, containing C_+^o. So there existsa vector $x \in R^n$, such that $\forall p \in C_+^o: px = 0$. By 7.4 $x \in C_{++}^{oo} = C$ and $-x \in C_{++}^{oo} = C$, which is a contradiction.

8. Dual sets and dual cones.

For any set, not containing the origin in its closed convex hull, both the upper dual set and the upper dual cone are not empty. It was shown (property 7.4), that then Cl Cone $C = C_{++}^{oo}$. It is also true that the closed cone of the dual set, equals its dual cone (see fig 5).

Property 8.1

If $C_+^* \neq \emptyset$, then Cl Cone $C_+^* = C_+^o$.

Proof

$\forall x \in C: px \geq 1 \Rightarrow \forall x \in C: px \geq 1$, hence $C_+^* \subset C_+^o$ and since C_+^o is a closed cone, also Cl Cone $C_+^* \subset C_+^o$. Now let $p \in C_+^o$. Since
Cone $C_+^* \setminus \{0\} = \{p | \exists \lambda > 0, \forall x > 0: \lambda\ px \geq 1\} =$
$\{p | \exists \eta > 0, \forall x \in C: px > \eta\}$, we have for any $q \in$ Cone C_+^* and $q \neq 0$: if $0 \leq \alpha < 1$, then $\alpha p + (1-\alpha) q \in$ Cone C_+^*.
Since Cl Cone C_+^* is closed, also $p \in$ Cl Cone C_+^*.

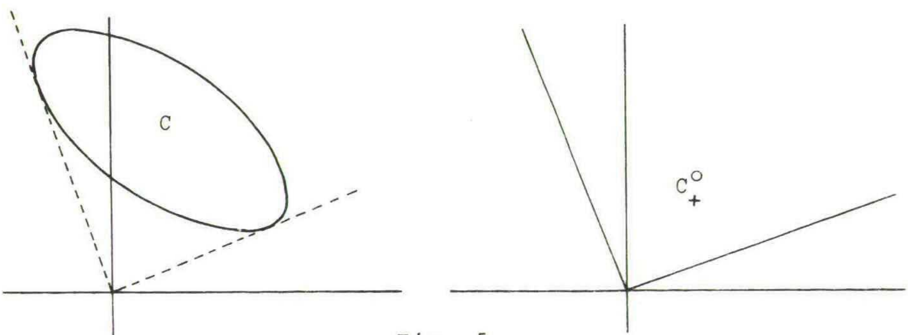


Fig. 5

<u>Corrollary</u> For any family C_i

$$\text{Cl Cone } (\cup C_i)^*_+ = \cap \text{ Cl Cone } C^*_{i+}$$

<u>Proof</u>

By 7.5: $(\cup C_i)^o_+ = \cap C^o_{i+}$

By 8.1: $\text{Cl Cone } (\cup C_i)^*_+ = (\cup C_i)^o_+$

by 8.1: $\text{Cl Cone } C^*_{i+} = C^o_{i+}$

and now the corrolary follows.

Similarly lower dual sets and lower dual cones are related. Instead of property 8.1, we get the two properties 8.3 and 8.4.

<u>Property 8.3</u>

$\text{Coneint } C^*_- = C^o_-$

<u>Proof</u>

$\text{Coneint } C^*_-\setminus\{0\} =: \{p \,|\, \forall x \in C, \forall \lambda > 0: \lambda\ px \leq 1\}$

Let $p \in C^o_-$, hence $\forall x : px \leq 1$, and so $\forall x \in C$, $\forall \lambda > 0$:

$\lambda\ px \leq 0 < 1$, hence $p \in \text{Coneint } C^*_-$.

If $p \notin C^o_-$, then for some $x_o \in C$: $px_o = \alpha > 0$. Choose

$\lambda > \frac{1}{\alpha} > 0$, then $\lambda\ px > \frac{1}{\alpha}\ px_o = 1$, hence $p \notin \text{Coneint } C^*_-$.

Note that for C, such that $0 \in \text{Int } C$, $C^o_- = \emptyset$ and C^*_- is compact.

<u>Property 8.4</u>

If C is a type S set: $\text{Cl Cone } C^*_- = (\text{Coneint } C)^o_-$.

<u>Proof</u>

Replace C by C^*_- in 8.3: $\text{Coneint } (C^*_-)^*_- = (C^*_-)^o_-$.

Since C is type S: $\text{Coneint } C^{**}_{--} = \text{Coneint } C$.

By prop. 7.4: $(C^*_-)^o_- = (\text{Cl Cone } C^*_-)^o_-$, hence $\text{Coneint } C =$

$(\text{Cl Cone } C^*_-)^o_-$. By taking dual cones $(\text{Coneint } C)^o_- =$

$(\text{Cl Cone } C^*_-)^o_-$.

Note that for C compact, $0 \in \text{Int } C^*_-$ and hence $\text{Cl Cone } C^*_- =$ R^{n*}.

Remark

Our concept of dual cone, should be distinguished from another (nearly related) concept, also called dual (or polar) cone: $F(C) = \{p,\lambda \mid x \in C: px \leq \lambda\}$. Now C^*_- is the projection of $F(C) \cap \{p,\lambda \mid \lambda = 1\}$ on R^{n*} and C^o_- is the projection of $F(C) \cap \{p,\lambda \mid \lambda = 0\}$ on R^{n*}.

9. Dual summation.

Let C_i $(i=1,2,\ldots,n)$ be a finite number of type A sets, such that their sum ΣC_i is also a type A set, which is true, if the sum of their dual cones is pointed. (theorem 3.4).
How to express the dual of the sum $(\Sigma C_i)^*$ in terms of C^*_i?
In [13] we proved that

$$(\Sigma C_i)^*_+ = \mathrm{Cl}\ \{p \in R^{n*}/\exists\alpha_i > 0, \exists p_i \in C_i : p = \alpha_i p_i \text{ and } \Sigma\alpha_i = 1\}.$$

The operation between the braces is called inverse addition in [6], where $\alpha_i = 0$ is not excluded.
Now it is easy to see that the right hand term is equal to $\mathrm{Cl} \bigcup_{A_+} \bigcap_I \alpha_i C^*_i$ for $A_+ = \{\alpha_i \in R \mid \alpha_i > 0,\ \Sigma\alpha_i = 1\}$:

$$\{p \mid \exists\alpha_i \in A_+, \exists p_i \in C^*_i,\ \forall_i : p = \alpha_i p_i\} =$$

$$\{p \mid \exists\alpha \in A_+, \forall_i : p \in \alpha_i\ C^*_i\} =$$

$$\{p \mid \exists\alpha \in A_+, \forall_i : p \in \cap\ \alpha_i\ C^*_i\} = \bigcup_{A_+} \bigcap_I \alpha_i\ C^*_i$$

In def. 3.3 we defined OC as Cl Con C (for C type A).
Now it appears that we can allow α_i to be 0; we denote dual summation by $\boldsymbol{\Sigma}$ and $\boldsymbol{\oplus}$.

Theorem 9.1

If C_i $(i \in I = \{1,2,\ldots,n\})$ are type A sets and if $(\bigcup_I \mathrm{Cl\ Cone}\ C_i) \cap -(\bigcup_I \mathrm{Cl\ Cone}\ C_i) \supset \{0\}$ then

$$(\Sigma C_i)^*_+ = \bigcup_A \bigcap_I \alpha_i\ C^*_i = \boldsymbol{\Sigma}\, C^*_i$$

for $A = \{\alpha \mid \alpha_i \geq 0,\ \Sigma\alpha_i = 1,\ i \in I\}$

Proof

Let $C^* = (\Sigma\, C_i)^*_+$ and $S = \bigcup_A \bigcap_I \alpha_i\, C^*_{i+}$

$S \subset C^*$: Let $p \in S$, then there exists $\alpha \in A$, such that $p \in \alpha_i\, C^*_{i+}$. For $\alpha_i > 0, p \in \frac{1}{\alpha_i}\, C^*_{i+}$ and for $\alpha = 0$, $p \in 0\, C^*_{i+} = \text{Cl Cone } C^*_{i+}$. Hence for all i, we have $x_i \in C_i \Rightarrow px_i \geq \alpha_i$. So for $x \in \Sigma\, C_i$:
$p\, \Sigma\, x_i = \Sigma\, p\, x_i \geq \Sigma\, \alpha_i = 1$ and $p \in (\Sigma\, C_i)^*_+$

$S \supset C^*$: We first show: Cl Cone $C^* = \cap$ Cl Cone C^*_{i+}:
By 8.1: Cl Cone $\cap\, C^*_{i+}$ = Cl Cone $(\cup\, C_i)^*_+$ and
Cl Cone $(\Sigma\, C_i)^*_+ = (\Sigma\, C_i)^o_+ = (\text{Cl Cone } \Sigma\, C_i)^o_+$,
Cl Cone $(\cup\, C_i)^*_+ = (\cup\, C_i)^o_+ = (\text{Conv Cl Cone } \cup\, C_i)^o_+$, whereas
Cl Cone $\Sigma\, C_i = \Sigma$ Cl Cone C_i = Conv Cl Cone $\cup\, C_i$.
Now let $p \in C^*_+$, so $\forall\, x \in \Sigma\, C_i : px \geq 1$. Since $p \in$ Cl Cone C^*_+, also $p \in \cap$ Cl Cone C^*_{i+}.
Therefore $\inf_{x \in C_i} px = \beta_i \geq 0$ and $\Sigma\, \beta_i \geq 1$. Choose $\alpha_i = \frac{\beta_i}{\Sigma\beta_i}$, now $\alpha_i \geq 0$ and $\Sigma\, \alpha_i = 1$. For $\alpha_i > 0 : p \in \beta_i C^*_{i+} \subset \alpha_i C^*_{i+}$, for $\alpha = 0, p \in 0\, C_{i+}$ = Cl Cone C^*_{i+}. Hence $p \in \cap\, \alpha_i\, C^*_{i+}$ and $p \in S$.

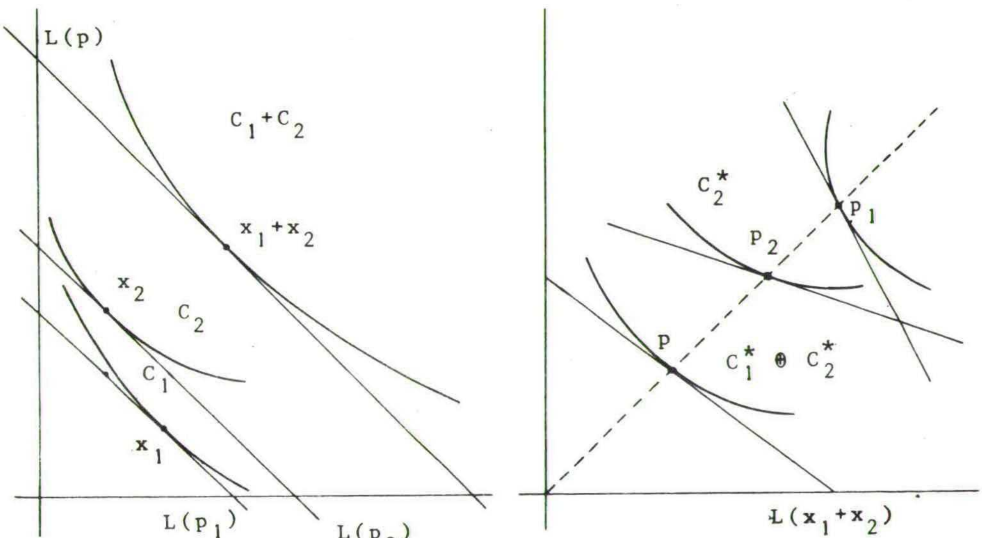


Fig. 6

Remark

From $(\Sigma\, C_i)^*_+ = \text{Cl} \bigcup_{A_+} \bigcap_I \alpha_i\, C^*_i$, it follows (for type A sets)

by appl.prop.7.5 $(\Sigma\, C_i)^{**}_{++} = \Sigma\, C_i = (\bigcup_{A_+} \bigcap_{I} \alpha_i\, C^*_{i+})^*_+ = \bigcap_{A_+} \text{Conv} \bigcup_{I} \frac{1}{\alpha_i} C_i$
for $A_+ = \{\alpha\} \alpha_i > 0\ \alpha\ \Sigma\alpha_i = 1\}$.
Not that for $x = \Sigma\, x_i$ we have for any $\alpha > 0$, $x = \Sigma\, \alpha_i (\frac{1}{\alpha_i} x_i)$.

10. Separation and intersection properties of a type A and a type S set.

If C and Y are two convex sets, there are four cases:

- they intersect in their (relative) interiors
- they intersect in their boundaries, not in their interiors, they "touch": now a hyp. L(p) separates both sets and does support them in their intersection.
- they do not intersect, but touch asymptotically: in this case they can be separated by a hyperplane, which is an asymptotic support of at least one of the sets; now any parallel hyperplane intersects one of the sets in its relative interior.
- they do not intersect, and do not touch asymptotically, and they are strictly separated by a hyperplane.

In the rest of this section, we consider the intersection properties of two sets, one being a type A set and the other a type S set, and their duals, i.e. the upper dual of the type A set and the lower dual of the type S set.
Given certain assumptions, we can state (theorem 10.3b)

> If and only if the two sets are disjoint, their duals intersect in their (relative) interiors (and the same holds when the dual and the original sets are interchanged).

and (theorem 10.3c)

> If and only if the two sets touch, their duals touch.

It is the possibility of asymptotic touching, whichspoils these simple properties for the general case.
We now first give two lemma's necessary for the proof of theorem 10.3, and which allow to formulate conditions, which exclude "asymptotic" touching.

Note that if Y is a type S set, then for $\lambda < 1$, λD is in the relative interior of Y with respect to its closed cone.

Lemma 10.1

If C is a type A set and Y is a type S set, and if Cl Cone C $\cap$ Coneint Y $\subset \{0\}$, then
a) $C \cap Y$ is compact
b) $C \cap Y = \emptyset \Rightarrow \exists \lambda > 1: C \cap \lambda Y = \emptyset$.

Proof

a) In [2], 1.9 (9), Debreu gives this property for Asc C $\cap$ - Asc Y $\subset \{0\}$ and a) follows from prop. 1.1.

b) Assume $C \cap \mu Y \neq \emptyset$ for some $\mu > 1$. This intersection is compact, since Coneint Y = Coneint μ Y. Hence $C \cap \mu Y$ and Y are strictly separated by some hyperplane L(p).
Let $\alpha = \min \{px | x \in C \cap \mu Y\}$; now $1 < \alpha < \mu$. Choose $1 < \lambda < \alpha$. Now $L(\frac{1}{\lambda}p)$ strictly separates λY and $C \cap \mu Y$.
Since $C \cap \lambda Y \subset C \cap \mu Y$, and $\lambda Y \cap (C \cap \mu Y) = \emptyset$, we have $C \cap \lambda Y = \emptyset$.

Lemma 10.2

If C is a type A set and Y is a type S set,
Cl Cone C $\cap$ Coneint Y $\subset \{0\} \iff$ Cl Cone C^*_+ and Cl Cone Y^*_- cannot be separated by a hyperplane.

Proof

By properties 8.1 and 8.3,
Cl Cone $C^*_+ = C^o_+$ and Cl Cone $Y^*_- = (\text{Coneint } Y)^o_-$; $\Rightarrow$ Assume the left hand side of the implication is true, but that the dual cones can be separated, i.e. for some
$x \neq 0$: $\forall p \in C^o_+ : px \geq 0$ and $\forall p \in (\text{Coneint } Y)^o_- : px \leq 0$, so
$x \in C^{oo}_{++}$ = Cl Cone C and $x \in (\text{Coneint } Y)^{oo}_{--}$ = Coneint Y,
and that is a contradiction.
$\Leftarrow$ Let $0 \neq x \in$ Cl Cone C $\cap$ Coneint Y. Now
$\forall p \in$ Cl Cone $C^*_+ : px \geq 0$ and $\forall p \in$ Cl Cone $Y^*_- : px \leq 0$. Hence L(x) separates the two sets.

Theorem 10.3

Let C be a type A set and Y a type S set.

a) $C \cap Y = \emptyset$ and $\nexists\lambda > 1$: $C \cap \lambda Y = \emptyset \iff$

$$C_+^* \cap Y_-^* \neq \emptyset \text{ and } \exists\mu < 1: C_+^* \cap \mu Y_-^* \neq \emptyset$$

b) If Cl Cone C and Coneint $Y \subset \{0\}$ (or equivalently, if Cl Cone C_+^* and Cl Cone Y_-^* cannot be separated by a hyperplane), then

$$C \cap Y = \emptyset \iff C_+^* \cap Y_-^* \neq \emptyset \text{ and } \exists\mu < 1: C_+^* \cap \mu Y_-^* \neq \emptyset$$

c) If Cl Cone C and Coneint $Y \subset \{0\}$ and Cl Cone C and Cl Cone Y cannot be separated by a hyperplane, then

$$C \cap Y \neq \emptyset \text{ and } \forall\lambda < 1: C \cap \lambda Y = \emptyset \iff$$
$$C_+^* \cap Y_-^* \neq \emptyset \text{ and } \forall\mu < 1: C_+^* \cap \mu Y_-^* = \emptyset.$$

Note that we may replace C and Y by C_+^* and Y_-^* and C_+^* and Y_-^* by C and Y, in a) and b) getting the "dual" version of a) and b).

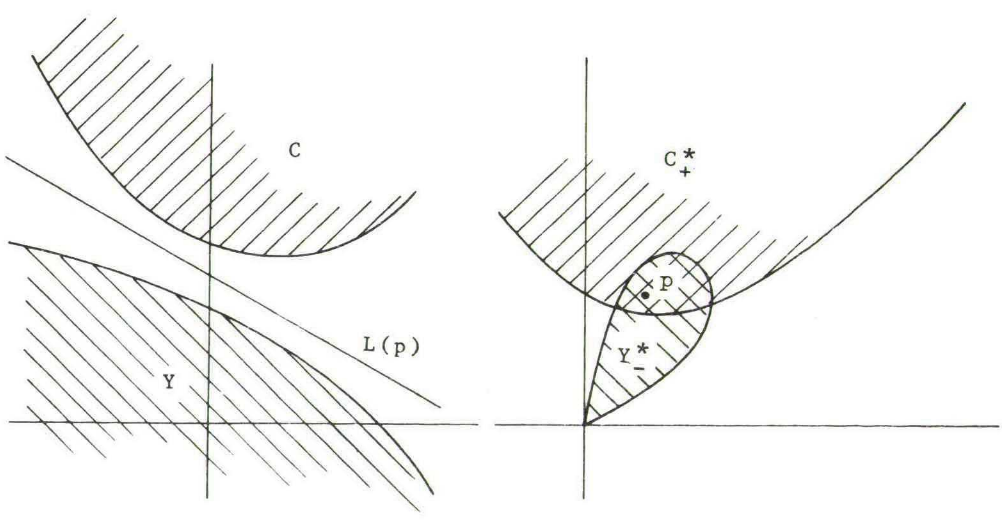


Fig. 7

Proof.

a) $\Rightarrow$. Since C and Y do not intersect, there exists some hyperplane L(p) separating both sets and now $p \in C_+^* \cap Y_-^*$. By the same argument there exists some $q \in C_+^* \cap (\lambda Y)_-^*$. Since $(\lambda Y)_-^* = \frac{1}{\lambda} Y_-^*$, $C_+^* \cap \mu Y_-^* \neq \emptyset$ for $\mu = \frac{1}{\lambda}$.
$\Leftarrow$ There exist p and $\mu < 1$, such that $p \in C_+^* \cap Y_-^*$ and $\mu p \in C_+^* \cap \mu Y_-^* \subset C_+^* \cap Y_-^*$. Hence $\forall x \in C: px > \mu px \geq 1$ and $\forall x \in Y: \mu px < px \leq 1$, or $\mu px = px = 0$. Choose α, such that $\mu < \alpha < \frac{1+\mu}{2} < 1$ and $\lambda = \frac{2}{1+\mu} > 1$. Now $L(\alpha p)$ strictly separates C and λY and therefore also C and Y, so $C \cap Y = \emptyset$ and $C \cap \lambda Y = \emptyset$.

b) Follows directly from lemma 10.2b, since $C \cap Y = \emptyset \Rightarrow C \cap \lambda Y = \emptyset$ for some $\lambda > 1$.

c) $\Rightarrow$ Suppose $C_+^* \cap Y_-^* = \emptyset$. Then by b), (after interchanging original and dual sets): $\exists \lambda < 1: C_+^* \cap \lambda Y_-^* \neq \emptyset$, which is a contradiction. If $C_+^* \cap \lambda Y_-^* \neq \emptyset$ for some $\lambda < 1$, then by a), $C \cap Y = \emptyset$, which is also a contradiction. The converse follows by interchanging dual and original sets.

11. Continuity of dual correspondences.

Let $C: S \to R^n$ be a correspondence.
We call dual correspondence: $C_+^*: S \to R^{n*}$, where, for $s \in S$

$$C_+^*(s) = [C(s)]_+^*$$

We discuss in this section correspondences, such that every set C(s) is closed, convex and $0 \notin C(s)$, so we do not require C(s) to be aureoled, however obviously any $C^*(s)$ is a type A set. It will be shown that lower-hemi continuity (closedness) of C, imply closedness (lower hemi continuity) of C^*. This implies that the correspondence Au $C: S \to R^n$, when Au C(s) are the aureoled **closures** of C(s), has the same continuity as C. Finally continuity properties also hold for dual sums.

Remark.

In [9] the term dual correspondence is used in a different sense. There it denotes $F^*: R^{m*} \to R^{n*}$, dual to a correspondence $F: R^n \to R^m$, and can be considered as a generalized adjoint.

We use the following continuity definitions (see [1])

- closedness: if $s^t \to s^o$, $x^t \to x^o$ and $x^t \in C(s^t)$, then $x^o \in C(s^o)$
- upper hemi-continuity: C is closed and C(s) is compact
- lower hemi-continuity: if $s^t \to s^o$ and $x^o \in C(s^o)$, then there exists a sequences $x^t \to x^o$, such that $x^t \in C(s^t)$; or equivalently: if A an open set and $C(s^o) \cap A \neq \emptyset$, then there exists some neighbourhood U of s^o, such that $s \in U \Rightarrow C(s) \cap A \neq \emptyset$.
- continuity: if C is both l.h.c. and u.h.c.

Theorem 11.1

Let $C:S \to R^n$ be a correspondence and C^* its dual such that for all s, C(s) is a closed, convex set and $0 \notin C(s)$.

1) If C is l.h.c., then C^* is closed
2) If C is closed and if for some ε, $C^*(s) \cap B_\varepsilon(0) = \emptyset$ for all s, then C^* is l.h.c.

Proof.

a. Closedness: for $s^t \to s^o$, $p^t \in C^*(s^t)$, $p^t \to p^o$, it has to be shown that $p^o \in C^*(s^o)$. Suppose $p^o \notin C^*(s^o)$. Then there exists some $x^o \in C(s^o)$, such that $p^o x^o = 1-\alpha$ (for $0<\alpha<1$). By l.h.c. of C, there exists a sequence $x^t \to x^o$, such that $x^t \in C(s^t)$. Choose t_1 and t_2, such that: $t > t_1 \Rightarrow |p^t - p^o| < \min\left(\frac{1}{3}\alpha \;\; \frac{\frac{1}{3}\alpha}{|x^o|}\right)$ and $t > t_2 \Rightarrow |x^t - x^o| < \min\left(\frac{1}{3}\alpha \;\; \frac{\frac{1}{3}\alpha}{|p^o|}\right)$. For $t > t_2$ and $t > t_1$:

$$p^t x^t = [p^o + (p^t - p^o)][x^o + (x^t - x^o)] =$$

$$p^o x^o + p^o(x^t - x^o) + (p^t - p^o)x^o + (p^t - p^o)(x^t - x^o) <$$

$$(1-\alpha) + |p^o| \frac{\frac{1}{3}\alpha}{|p^o|} + \frac{\frac{1}{3}\alpha}{|x^o|}|x^o| + \frac{1}{9}\alpha^2 = 1 - \frac{1}{3}\alpha - \frac{1}{9}\alpha^2 < 1$$

Since $x^t \in C(s^t)$, this implies that $p^t \notin C^*(s^t)$ for t sufficiently large, and that is a contradiction.

b) <u>l.h.c.</u>: we first prove the following lemma:

<u>Lemma:</u> If $0, p^o \in R^n$, $B_\varepsilon(p^o)$ and $B_\eta(p^o)$ are open neighbourhoods of 0 and p^o, then there exists an open convex set D, such that $D \subset B_\varepsilon(0) \cup \mathrm{St}\, B_\eta(p^o)$.

<u>Proof of the lemma</u>: Choose $\phi = \frac{\varepsilon\eta}{|p|+\eta}$ and $D = \{q \mid q = \lambda p,\ 0 \le \lambda \le 1\} + B_\phi(0)$. This set D is <u>convex</u> and <u>open</u>. Let $q \in D$. Now $q = \lambda p + \phi z$, for $|z| \le 1$ and $0 \le \lambda \le 1$.

For $\lambda \ge \frac{\varepsilon}{|p|+\eta}$, we have $q = \lambda(p + \frac{1}{\lambda}\phi z) = \lambda(p+\rho z)$, for $\rho = \frac{1}{\lambda}\phi \le \frac{|p|+\eta}{\varepsilon}\,\frac{\varepsilon\eta}{|p|+\eta} = \eta$.

Since $p+\rho z \in B_\eta(p)$, we have $\lambda(p+\rho z) \in B_{\lambda\eta}(\lambda p) \subset \mathrm{St}\, B_\eta(p)$.

For $\lambda < \frac{\varepsilon}{|p|+\eta}$, it follows $|\lambda p + \rho z| \le \lambda|p| + \phi < \frac{\varepsilon}{|p|+\varepsilon}|p| + \frac{\varepsilon\eta}{|p|+\varepsilon} = \varepsilon$. Hence $q \in B_\varepsilon(0)$.

<u>Proof</u> of the theorem

It is to be shown that $s^t \to s^o$ and $p^o \in C^*(s^o)$ implies the existence of $p^t \to p^o$ and $p^t \in C^*(s^o)$.

Suppose this is not true. Then there exists a subsequence $s^v \to s^o$ and an $\eta > 0$, such that:

$$\forall v: C^*(s^v) \cap B_\eta(p^o) = \emptyset$$

Since $C^*(s^v)$ is aureoled, this implies $C^*(s^v) \cap \mathrm{St}\, B_\eta(p^o) = \emptyset$.

For some subsequence s^r, we have by assumption:

$$C^*(s^r) \cap B_\varepsilon(0) = \emptyset$$

Hence for the set D, as defined in the lemma, $C^*(s^r) \cap D = \emptyset$.

Let $E \subset D$ be a closed set containing 0 and p^o.

E is also compact. Obviously $C^*(s^v) \cap E = \emptyset$.

Further there must exist some $\mu > 1$, such that $\mu p^o \in C^*(s^o) \cap D$. Now $C(s^r) \cap L(\mu p^o) \ne \emptyset$ and $E^*_- \cap L(p^o)$ is not empty and compact. This implies:

$$C(s^r) \cap L(\mu p^o) \cap E^*_- \subset C^{**}(s^r) \cap L(\mu p^o) \cap E^*_-$$

Hence

$$C(s^r) \cap L(\mu p^o) \cap E^*_- \subset E^*_- \text{ and compact.}$$

Therefore there exists a sequence $x^v \in C(s^v) \cap L(\mu p^o) \cap E^*_-$ which contains by the compactness of E^*_- a convergent subsequence $x^w \to x^o$. By the closedness of C, we have $x^o \in C(s^o)$ and also $x^o, \frac{1}{\mu} x^o \in C^{**}(s^o)$, since the last set is aureoled. Since $C^{***}(s^o) = C^*(s^o)$, applying theorem 10.3, we get $C^*(s^o) \cap D = \emptyset$. This is a contradiction.

<u>Theorem 11.2</u>

Let $X \supset V$, both being closed and convex and $0 \in X$.
Let $C: S \to X$ be closed and l.h.c. and $C^*(s) \supset V$ for all s.
Then Conv Au $C = C^{**}: S \to X$ is closed and l.h.c.

<u>Proof</u>

C^* is closed, l.h.c., since $C^*(s) \cap B_\varepsilon(0) = \emptyset$ for some $\varepsilon > 0$, since $C^*(s) \subset V^*$. Therefore $C^{**}(s)$ is closed and l.h.c. since $C^{**}(s) \subset X^{**} \not\ni 0$.

<u>Theorem 11.3</u>

Let C_i be a family of correspondences such that $C_i: S \to X_i$, closed and l.h.c. and $C_i(s)$ is a type A set for all s and X_i is also a type A set, where $\boldsymbol{\Sigma}\, X_i \not\ni 0$.
Now the correspondence $C = \boldsymbol{\Sigma}\, C_i$ is <u>closed</u> and <u>l.h.c</u>.

<u>Proof</u>

$C = \bigcup_A \bigcap_\alpha \alpha_i\, C_i(s)$ for $A = \{\alpha \mid \Sigma\alpha_i = 1, \alpha_i \geq 0\}$.
<u>l.h.c.</u>: Since C_i are l.h.c., $\alpha_i\, C_i(s)$ are l.h.c., so $\bigcap_I \alpha_i\, C_i(s)$ is l.h.c. (intersection of a finite number of correspondences, see [1]p.120), hence $\bigcup_\alpha (\bigcap_I C_i(s))$ is l.h.c. (union of a family of l.h.c. correspondences, see Berge p.119).
<u>Closedness</u>: by theorem 11.1, $C_i^*(s)$ is l.h.c. Obviously $\Sigma\, C_i^*(s)$ is l.h.c. Therefore $[\Sigma\, C_i^*(s)]^* = \boldsymbol{\Sigma}\, C_i(s)$ is closed, by theorem 11.1.

<u>Corrollory</u>

From this theorem the closedness of the ordinary sum $\Sigma\, C_i(s)$ can be derived.

PART II

12 Definition of the economy.

We distinguish a commodity space R^n and a price space R^{n*}. For any $x \in R^n$ and $p \in R^{n*}$, the inner product

$$px = \sum_1^n p^i x^i$$

represents an amount of money.

The economy is defined by the following concepts:

1. A total production set $Y \subset R^n$ of all possible input output combinations in the economy.
2. The set $I = \{1,2,\dots,n\}$ of consumers.
3. An income distribution $\lambda_i(p)$, which assighns to the i'th individual a fraction $\lambda_i(p)$ of the value py of the optimal production y at price $p \in R^n$. It is defined for all p, such that $\max_{y \in Y} py$ exists. Obviously $\sum_I \lambda_i(p) = 1$.
4. A consumption set $X_i \subset R^n$ for each $i \in I$.
5. A preference relation $\succsim_i$ on X_i for each $i \in I$.

The production set Y may be considered (see [3]) as the sum of a technological production set Z and a vector of primary resources: $Y = Z + \{w\}$. In this case $Z = \Sigma Z_j$, where Z_j is the production set of the j'th producer, and $w = \Sigma w_i$, where w_i is the vector of resources owned by $i \in I$.

The income could possible be split up into two parts: the value of primary resources owned by i and his part $\xi_i(p)$ in net profit. In this case

$$\lambda_i(p) = \frac{p\,w_i + \xi_i(p)(py-pw)}{p\,y} \text{ where } \sum_I \xi_i(p) = 1.$$

Neither Z nor w occur explicitely in this paper.

Definition 12.1

A competitive equilibrium is an allocation $\bar{x}_i \in X_i$, a production vector $\bar{y} \in Y$ and a price vector $\bar{p} \in R^n$, such that:

$$\forall z \in Y : \bar{p}\,\bar{y} \geqq \bar{p}\,z$$

$$\forall i \in I : \bar{p}\,\bar{x}_i = \lambda_i(\bar{p})\bar{p}\,\bar{y}$$

$$\forall i \in I : z \succ_i \bar{x}_i \Rightarrow \bar{p}\,z > \bar{p}\,\bar{x}_i$$

$$\sum_I \bar{x}_i = \bar{y}$$

13. The preference correspondence.

The preference relation can also be represented by a correspondence, called preference correspondence $C_i : X_i \to X_i$, where

Definition 13.1

$C_i(x) = \{y \in X_i | y \succsim_i x\}$ for $x \in X_i$.

We have $z \succ_i y \iff z \in C_i(y)$ and $y \notin C_i(z)$, and $z \sim y \iff z \in C_i(y)$ and $y \in C_i(z)$.
An allocation is an n-tuple $x_i (i=1,2,\ldots,n)$ such that $x_i \in X_i$.
A feasible allocation is an allocation such that $\Sigma\, x_i \in Y$.
The set $\Sigma\, X_i \cap Y$ contains all vectors x, such that x corresponds to a feasible allocation, i.e. x can be divided among the consumers such that $\Sigma\, x_i = x$ for $x_i \in X_i$. An $x_i \in X_i$, which is a component of such a feasible allocation, is a feasible consumption for the i'th consumer and

$$F_i = \{x_i \in X_i | [\sum_{j \neq i} X_j + \{x_i\}] \cap Y \neq \emptyset\}$$

is the set of feasible consumptions of i. We define a set V_i of non-feasible consumptions, namely the set of non-feasible consumptions that are strictly preferred to all feasible consumptions.

Definition 13.2

$V_i = \{v \in X_i | \forall x_i \in F_i : v \succ_i x_i\}$

If $x_i \in F_i$, we have $V_i \subset C_i(x_i) \subset X_i$.
The equilibrium of definition 12.1 can also be expressed in terms of the preference correspondence. If we assume that $\lambda_i(p) > 0$ for all $i \in I$ and $py > 0$, and if we normalize prices in such a way that $\bar{p}\,\bar{y} = 1$, then for the equilibrium as defined, holds:

$L(\bar{p})$ supports Y in $\bar{y}$

$\bar{y} \in L(\bar{p}) \cap \Sigma\, C_i(\bar{x}_i)$

$\bar{x}_i \in L(\bar{p}_i) \cap C_i(\bar{x}_i)$ for $\bar{p}_i = \dfrac{1}{\lambda_i(\bar{p})}\,\bar{p}$

$\bar{y} = \Sigma\, \bar{x}_i$

$\bar{z} \in L(\bar{p}_i) \Rightarrow z \notin C(x)$ or $x \in C(z)$

Obviously $\bar{x}_i \in X_i \setminus V_i$

14. Representation of the economy in the price space.

The economy defined in section 12, can, by taking dual sets, also be represented in the price space. This representation could be considered as "equivalent", if the original economy can be reconstructed by taking duals of duals. In that case no information is lost. However with respect to certain problems as e.g. equilibrium as discussed below, it is sufficient if all "relevant" information is preserved, which means in the case of equilibrium, that any equilibrium in the commodity space corresponds to an equilibrium in the price space and conversely.
In the dual economy we distinguish two types of prices: individual prices and general prices. This distinction parallels the distinction in the commodity space, where there are

- individual consumption bundles x_i for each i
- a total consumption $x = \Sigma\ x_i$
  Total consumption is derived from individual consumptions by summation (both of vectors and of sets). Only total consumption is directly comparable with (total) production.

In the price space, we have

- individual prices p_i for each $i \in I$
- general prices p, where $p = \lambda_i p_i$.

General prices hold for <u>total</u> consumption and for (total) production. They are chosen so that the value of total consumption and production equals 1, hence they are expressed with total income as unit: if $\tilde{p}$ is a vector of prices expressed in florins and M the income of the economy in florins, then $p = \frac{1}{M}\,\tilde{p}$. Individual prices are chosen so that the value of individual consumption equals 1, hence their unit is the individual's income: for $M_i = \lambda_i M$ the individual's income in florins, $p_i = \frac{1}{M_i}\,\tilde{p} = \frac{1}{\lambda_i}\,p$. General prices are derived from individual prices by the operation of dual summation (see section 9) both for vectors and sets. We have

$$p = \lambda_i\ p_i \text{ with } \lambda_i > 0 \text{ and } \Sigma\ \lambda_i = 1.$$

The dual economy is defined below, without any assumptions, so that no preservation of properties is guaranteed. Upper dual sets will be used for consumption, lower dual sets for production; therefore we shall generally omit the suffixes + and −.

1. $X_i^* = X_{i+}^* = \{p_i \in R^{n^*} \mid x \in X_i : p_i x \geq 1\}$ is the set of all individual prices of i, such that any consumption $x \in X_i$ costs at least 1. Hence prices of X_i^* are either <u>impossible</u> (if $p_i\ x > 1$ for all x) or <u>just possible</u> (if $p_i\ x = 1$ for some x), provided that the consumer's income is equal to 1. In the latter case some boundary point of X_i is available. If $p_i \in \text{Int}\ X_i^*$, then p_i is impossible, since

$p_i\ x > 1$ for all x (the converse is not true). Obviously $X_i^* \neq \emptyset \iff 0 \notin X_i$, and $X_i^{**} = \text{Cl Conv Au } X_i$. So if $0 \in X_i$, all information is lost, otherwise only the smallest type A set, containing X_i is preserved.

2. $V_i^* = V_{i+}^* = \{p_i \in R^{n*} | \forall x \in V_i : p_i\ x \geq 1\}$ contains all individual prices p_i, such that any consumption from V_i costs at least 1. Obviously any price, corresponding to a feasible consumption must have this property. Hence for an equilibrium price holds $p_i \in V_i^* \setminus \text{Int } X_i^*$. Note that $X_i^* \subset V_i^*$. We may consider V_i^* as the set of <u>feasible</u> prices for i.

3. $C_i^*(x_i) = [C(x_i)]_+^*$ is the set of all prices, such that any commodity bundle preferred or indifferent to x, costs at least 1. So such a bundle is not or just available at such a price (the income being equal to 1). Obviously (see section 5), $C_i^*(x) \neq \emptyset \iff 0 \notin C_i(x)$ and $C_i^{**}(x) = \text{Cl Conv Au } C_i(x)$.
The correspondence C_i^* maps X_i into R^{n*}. If we restrict C_i to $X_i \setminus V_i$, then we have: $C_i^* : X_i \setminus V_i \to V_i^*$.

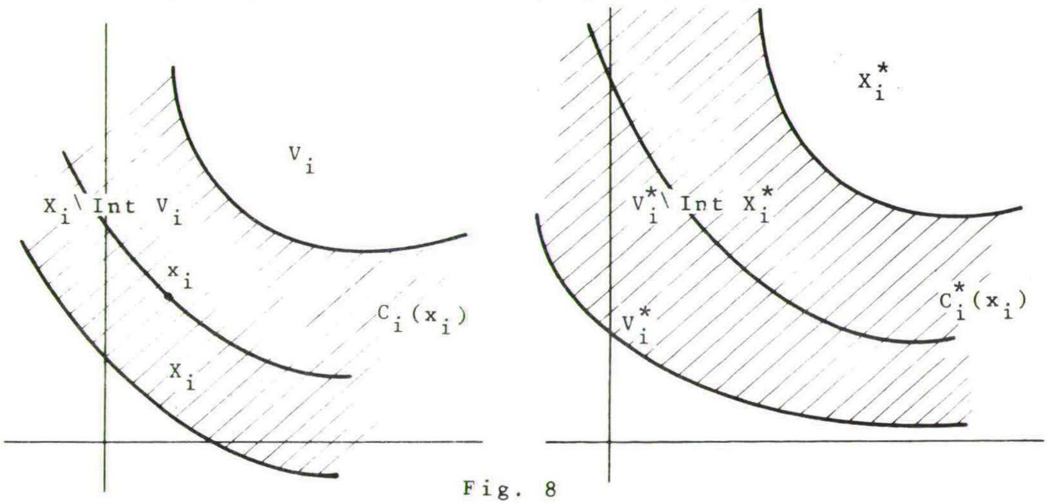


Fig. 8

From C_i^* we derive a correspondence $\hat{C}^* : V_i^* \setminus \text{Int } X_i^* \to V_i^*$, where

if $p \in V_i^* \setminus X_i^*$, $\hat{C}^*(p) = \bigcap_{R(p)} C_i^*(x)$ for $R(p)=\{p | p \in C_i^*(x)\}$

if $p \in \text{Bnd } X_i^*$, $\hat{C}_i^*(p) = \bigcup_{R(p)} C_i^*(x)$ for $R(p)=\{p | p \in \text{Bnd } C_i^*(x)\}$.

4. $Y^* = Y_-^* = \{p \in R^{n^*} | \forall y \in Y : py \leq 1\}$ is the set of all general prices, such that, no commodity bundle from Y costs more than 1. Vectors such that $px=1$ for some $y \in Y$, are on the boundary of Y^*. Obviously $Y^* \neq \emptyset$ and $0 \in Y^*$. $Y^{**} = \text{Cl Conv } \{\{0\}, Y\}$. So only points on the boundary of Y, that are also on the boundary of Y^{**}, are preserved as boundary points of Y.

In summary, the dual representation of the economy is defined by the following concepts, discussed above.
We introduced V_i at once, however it would have been possible to derive it, as we did for the original economy. For this economy an <u>equilibrium</u> is defined in definition 16.4.
This dual equilibrium is a point where the dual production set and a dual sum of dual preference sets, touch, exactly as in an equilibrium the production and a sum of preference sets touch. (see fig. 10).

<u>Concepts in the dual space</u>.

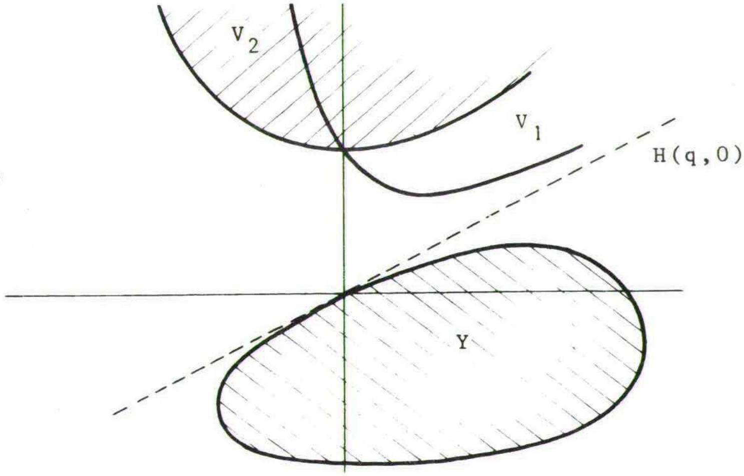
1. Y^*, dual total production set (set of impossible or just possible prices).
2. $I = \{1,2,\ldots,n\}$, set of consumers.
3. $\lambda_i : \text{Cone } Y^* \to R$, income distribution function.
4. X_i^*, set of impossible or just possible prices for $i \in I$.
5. V_i^*, set of feasible prices for $i \in I$.
6. $C_i^* : V_i^* \setminus \text{Int } X_i^* \to V_i^*$ correspondence associating worse and equivalent prices to each (individual) price.

Note that in these concepts commodities do not explicitely occur.

15. The assumptions.

The assumptions with respect to the consumer are stronger than the usual ones. It is required, that the consumption set does not contain the origin (b1). Assumption c1 is expressed in terms of closed cones and not in terms of asymptotic cones. Equivalence classes with non-empty interiors are excluded, unless such a class contains all "best" commodity bundles (the case of satiation).The assumption b5 is largely technical. Further it is required that all consumers have a strictly positive income at all feasible prices (d 2)(That total income py is strictly positive follows som c6).
However if some of these assumptions are not fullfilled, it might be possible to transform the economy by choosing a different origin such that the assumptions are true.
The assumptions with respect to production sets are rather weak. The (total) production set needs not contain the origin, however the set St Y $\cap$ Σ X should be equal to the intersection of the closed convex hull of St Y and Σ X(c4). Assumption c3 requires, that for some individual, there is a non-feasible consumption $x_i \in V_i$, which lays on a ray from the origin on which non-zero production is possible. Figure 9 depicts a situation, which is excluded.

Fig. 9



Note that c3 always holds if $0 \in \text{Int } Y$.
The assumptions with respect to the income distribution must hold for those prices that are feasible both with respect to production and consumption. This means that at such a price, production must have a maximum value and that at such a price and a positive income, no budget set may intersect the set V_i of non feasible consumptions. Hence the assumptions hold for

$$p \in \text{Cone } Y^* \cap \bigcap_I \text{Cone } V_i^* = P$$

Assumption d5 requires that at any feasible price an interior point of X_i is available.

<u>Assumptions</u> (in the commodity space).

b_1 X_i is closed and convex, $0 \notin X_i$

b_2 $\succsim_i$ is a preordering

b_3 $\forall x_i \in X_i : C_i(x_i)$ and $\{y \in X_i | x_i \succsim_i y\}$ are closed and if there exists y, such that $y \succ_i x_i$, then $x_i \in \text{Bnd } C_i(x_i)$.

b_4 $C_i(x_i)$ is convex for all $x_i \in X_i$.

b_5 If some hyperplane supports or asymptotically supports $C_i(x)$ and $C_i(y)$, and if it does not (asymptotically) support X_i, then $C_i(x) = C_i(y)$.

c_1 $[\Sigma \text{ Cl Cone } X_i] \cap -[\Sigma \text{ Cl Cone } X_i] = \{0\}$.

c_2 $[\Sigma \text{ Cl Cone } X_i] \cap \text{Coneint}[\text{Conv }\{Y,\{0\}\}] \subset \{0\}$

c_3 There exists no hyperplane $H(q,o)$, that separates Y and $\bigcup_I V_i$.

c_4 If there exists $x \in X_i$, such that for all $y \in X_i$, $x \succsim_i y$, then $[C_i(x_i) + \sum_{j \neq i} X_j] \cap Y = \emptyset$.

c_5 $\Sigma X_i \cap Y \neq \emptyset$

c_6 $\text{Cl Conv } Y \cap \Sigma X_i = \text{St } Y \cap \Sigma X_i$

d_1 $\lambda_i(p)$ is continuous for all i and for $p \in P$.

d_2 for $\mu > 0$ and $p \in P$: $\lambda_i(p) = \lambda_i(\mu p)$

d_3 $\sum_I \lambda_i(p) = 1$ for $p \in P$

d_4 $\lambda_i(p) > 0$ for $p \in P$

d_5 $\text{Int } X_i \cap \{x \mid p\, x_i \leq \lambda_i(p)[\max_{y \in Y} py]\} \neq \emptyset$ for all $p \in P$.

The assumptions with respect to the consumer are in terms of the preference relation $\succsim_i$. They imply for the preference correspondence:

<u>Theorem 15.1</u>

Given assumptions b1,b2,b3, and b4.

a. for all i and $x_i \in X_i : C_i(x_i)$ is closed and convex and $0 \notin C_i(x_i)$

b. $x_i \in \text{Bnd } C_i(x_i)$ or $\forall y \in X_i : x_i \in C_i(y)$

c. for all $x, y \in X_i : x \in C_i(y)$ or $y \in C_i(x)$

d. $y \in C_i(x) \Rightarrow C_i(y) \subset C_i(x)$.

e. the correspondence $C_i : X_i \to X_i$ is <u>closed</u> and <u>l.h.c.</u>

<u>Proof</u>.

a,b,c,d: obvious

e: <u>closedness</u>: let $x^t \to x^o$, $y^t \to y^o$ and $y^t \in C_i(x^t)$. Suppose $y^o \notin C_i(x^o)$. So $y^o \prec_i x^o$ and for some z, such that $x^o \succ z \succ y^o$, $x^o \in C_i(z)$ and $y^o \notin C_i(z)$.
So for some $t_x : t > t_x \Rightarrow x^t \in C_i(z)$ and for some t_y, $t > ty \Rightarrow y^t \notin C_i(z)$ and for $t > \max(t_x, t_y)$, $y^t \notin C_i(x^t) \subset C_i(z)$, which is a contradiction.

<u>l.h.c.</u>: Let A be an open set, such that $A \cap C(x^o) \neq \emptyset$. Let $y \in A$, such that $y \succ x^o$. (Since A is open and $C(x^o)$ is closed, there exists $y \in A$ and $y \notin \text{Bnd } C(x^o)$, so $y \succ x^o$). Now $\{x \mid y \succ x^o|$ is open and for all x of this set $y \in C(x)$ and $y \in A$.

For $x_i \in X_i$, the sum $\Sigma\, C_i(x_i)$ of preference sets is closed and convex. That it is closed follows from assumption c; since $\text{Asc } C_i(x_i) \subset \text{Cl Cone } C_i(x_i)$, we have

Σ Asc $C_i(x_i) \cap - \Sigma$ Asc $C_i(x_i) \subset \{0\}$ and by property 1.9 (9) in [3], sums are closed.
If $z \in$ Bnd $\Sigma C_i(x_i)$, then for some allocation $z, z = \Sigma z_i$ and $z_i \in$ Bnd $C_i(x_i)$. If $z \in$ Int $\Sigma C_i(x_i)$, then there exists an allocation z_i, such that $z_i \in C_i(x_i)$ for all i and for at least one i, $z_i \in$ Int $C_i(x_i)$, hence $z_i \succ_i x_i$.
These properties permit to define an equilibrium (see fig. 10a) for the economy merely by supporting hyperplanes, since interior points are always strictly preferred:

Theorem 15.2

Given b1, b2, b3, b4 and d2, $\bar{p}, (\bar{x}_i)$ and $\bar{y}$ are an equilibrium if

$$L(\bar{p}) \text{ supports } Y \text{ and } \Sigma C_i(\bar{x}_i) \text{ in } \bar{y} = \Sigma \bar{x}_i$$
$$L(\bar{p}_i) \text{ supports } C_i(\bar{x}_i) \text{ in } \bar{x}_i, \text{ for } \bar{p}_i = \frac{1}{\lambda_i(\bar{p})}\bar{p}$$
$$L(\bar{p}) \text{ supports } C_i(\bar{x}_i) \text{ in } z \Rightarrow z \in C_i(\bar{x}_i).$$

By the last statement a quasi-equilibrium is excluded: the case that $L(\bar{p}_i)$ still contains a better consumption x_i is ruled out; this could occur only on the boundary of X_i, $L(\bar{p}_i)$ supporting X_i in $\bar{x}_i$ and z.
The equilibria are not changed, if the preference sets are replaced by $C^{**}(x_i)$ = Cl Conv Au $C_i(x_i)$ = Au $C_i(x_i)$ and the production set is replaced by Y^{**} = Cl Conv $\{Y, \{0\}\}$.

Theorem 15.3

Given assumptions b1, b2, b3, b4, d4 and c6:
If and only if $\bar{y}\ (\bar{x}_i), \bar{p}$ is an equilibrium, it is also an equilibrium for the case that $C_i(x_i)$ is replaced by Au $C_i(x_i)$ and Y by Cl Conv $\{Y, \{0\}\}$.

Proof.

If $L(\bar{p})$ supports Y in $\bar{y}$, then it also supports Y^{**} in $\bar{y}$. Conversely, if $L(\bar{p})$ supports Y^{**} in y it also supports Y and since $Y^{**} \cap \Sigma X_i = Y \cap \Sigma X_i$, now $\bar{y} \in Y$. If $L(\bar{p}_i)$ supports $C_i(\bar{x}_i)$ in $\bar{x}_i$, it supports $C_i^{**}(\bar{x}_i)$ in $\bar{x}_i$, and conversely, since $C_i(\bar{x}_i)$ is convex, a point of $C_i^{**}(\bar{x}_i \setminus C_i(\bar{x}_i)$ cannot be supported by a hyperplane $L(p)$, but only by hyperplanes of the type $H(p,0)$.

Since we shall restrict the dual correspondence to the set $V_i^* \setminus \text{Int } X_i^*$, we have to show that $V_i \neq \emptyset$.

<u>Theorem 15.4</u>

Given assumption b1,b2,b3,b4,c1,c2,c4 and c5

$V_i \neq \emptyset$ for all i.

<u>Proof.</u>

If X_i contains a best point, i.e. a point x^o, such that $x^o \succ x_i$ for all $x_i \in X_i$, then by assumption c4, $[\sum_{j \neq i} X_j + C_i(x^o)] \cap Y = \emptyset$ and hence $x^o \in V_i$.
So let X_i not contain a best point, which by the continuity of $\succsim_i$ and the closedness of X_i, implies that X_i is not compact.
We first show:

$$\text{Int} \bigcap_I X_i^* \cap \text{Cone } Y^* \neq \emptyset \qquad \text{(i)}$$

By 5.5a, $\cap X_i^* = (\cup X_i)^* = (\text{Conv} \cup X_i)^*$.
$\text{Conv} \cup X_i \subset \Sigma \text{ Cl Cone } X_i$, so by assumption c1, $0 \notin \text{Conv} \cup X_i$.
By assumption c2, $\Sigma \text{ Cl Cone } X_i \cap Y$ is compact, hence for some $\mu < 1$, $(\text{Conv} \cup X_i) \cap \mu Y = \emptyset$ and so by theorem 10.3b

$$\text{Int } (\text{Conv} \cup X_i)^* \cap (\mu Y)^* \neq \emptyset$$

and $\text{Cone } (\mu Y)^* = \text{Cone } Y^*$, which proves (i). So we can choose some point $r \in \text{Bnd } Y^*$, such that $\mu r \in \text{Int} \cap X_i^*$ for some $\mu > 1$.
Now if there exists x_i, such that $r \in C_i^*(x_i)$, then certainly $r \in \text{Int } [\sum_{j \neq i} X_j^* \oplus C_i^*(x_i)]$ hence $[\sum_{i \neq j} X_j^{**} + C_i^{**}(x_i)] \cap Y = \emptyset$ and also $[\sum_{i \neq j} X_j + C_i(x_i)] \cap Y = \emptyset$.
So $C_i(x_i) \subset V_i$.
Hence it remains to prove that $r \in C_i^*(x_i)$ for some x_i, given that X_i is not compact. Since $\mu r \in \text{Int } X_i^*$, X_i^* and

the set $\{r\}$ cannot be separated by a hyperplane $H(q,0)$.
So the intersection of X_i^{**} and the upper dual set of r, $\{x \mid rx \leq 1\} = \{r\}_-^*$ is compact. So also $X_i \cap \{r\}_-^*$ is compact and this intersection certainly contains a best point x^o.
Since X_i is not compact and does not contain a best point, there exists a point $x_i \notin X_i \cap \{r\}_-^o$ such that $x_i \succ x^o$.
Hence $C_i(x_i) \cap L(r) = \emptyset$ and $r \in C_i^*(x_i)$.

16. The dual preference correspondence.

We restrict the correspondences C_i and C_i^* to $X_i \setminus \text{Int } V_i$. Since $C_i : X_i \setminus \text{Int } V_i \to X_i$ is l.h.c. and closed, $C_i^* : X_i^* \setminus \text{Int } V_i^* \to V_i^*$ is l.h.c. and closed, by theorem 11.1 (since $V_i^* \not\ni 0$). From C_i^* is derived a correspondence $\hat{C}_i^*$, mapping $V_i^* \setminus \text{Int } X_i^*$ into V_i^*. The set $\hat{C}_i^*(p_i)$ contains all prices, "equivalent or worse" then p_i, i.e. such that at such a price only commodity bundles can be bought, that are equivalent or worse then the best bundle available at the price p_i (and income 1).

Definition 16.1

If $p \in V_i^* \setminus X_i^* : \hat{C}_i^*(p) = \bigcap_{T(p)} C_i^*(x)$ for $T(p) = \{x \in X_i \mid p \in C_i^*(x)\}$
if $p \in \text{Bnd } X_i^* : C_i^{**}(p_i) = \bigcup_{T(p)} C_i^*(x)$ for $T(p) = \{x \in X_i \setminus \text{Int } V_i \mid p \in \text{Bnd } C_i^*(x_i)\}$

It will be shown below that for all p there exists some x, such that $\hat{C}_i^*(p) = C_i^*(x)$.
The properties of $\hat{C}_i^*$ are the same as those of C_i, as given in theorem 15.1.

Theorem 16.2

Given assumptions b1, b2, b3, b4 and b5:

a. $\hat{C}_i^*(p)$ is closed and convex and $0 \notin \hat{C}_i^*(p)$ for all $p \in V_i^* \setminus \text{Int } X_i^*$.

b. $p_i \in \text{Bnd } \hat{C}_i^*(p)$

c. for all $p,q \in V_i^* \setminus \operatorname{Int} X_i^*: p \in \hat{C}_i^*(q)$ or $q \in \hat{C}_i^*(p)$

d. $q \in \hat{C}_i^*(p) \Rightarrow \hat{C}_i^*(q) \subset \hat{C}_i^*(p)$

e. The correspondence $\hat{C}_i^*$ is closed and l.h.c. for $p \in V_i^* \setminus \operatorname{Int} X_i^*$

Proof.

a) directly follows from the definitions; b) is proved in lemma 16.3; c) and d) follow from the properties of dual sets and from assumptions b_2.

Before we prove the continuity, we first give a lemma. Note that only in the proof of e) assumption b_5 is used.

Lemma 16.3

For all $p \in V_i^* \setminus X_i^*$, $p \in \hat{C}_i^*(p)$ and there exists $x \in \operatorname{Int} X$, such that $\hat{C}_i^*(p) = C_i^*(x)$.

Proof.

Let $X^1 = \{x \in X_i \mid p^o \in C_i^*(x)\}$ and $X^2 = \{x \in X_i \mid p^o \notin C_i^*(x)\}$ for $p^o \in \operatorname{Int} X_i^*$. Obviously $X^1 \cup X^2 = X_i$ and $X^1 \cap X^2 = \emptyset$ and $X^1 \neq \emptyset$, since $p \in \operatorname{Int} X_i^*$.
We have $X^1 = \cup_{X^1} C(x) = \cap_{X^2} C(x)$: if $y \in X^1$, then $C(y) \subset X^1$; if $y \in \cup_{X^1} C(x)$, then $y \in X^1$; Obviously $X_1 \subset C(x)$ for $x \in X^2$.
Let $z^o \in \cap_{X^2} C(x)$. For $z^t = z^o$, $z^t \to z^o$. Choose $x^o \in \operatorname{Bnd} X^1 \cap \operatorname{Bnd} X^2$ and $x^t \to x^o$, for $x^t \in X^2$. Now for all t, $z^t = z^o \in C(x^o)$. So by the closedness of C, $z^o \in C(x^o)$, which proves $X^1 \supset \cap_{X^2} C(x)$. So X^1 is closed.
Now $\hat{C}^*(p^o) = \cap_{X^1} C^*(x) = (\cup_{X^1} C(x))^* = X^{1*}$.
Since for any x,y, $C(x) \subset C(y)$ or $C(x) \supset C(y)$, we also have $\hat{C}^*(p^o) = X^{1*} = (\cap_{X^2} C(x))^* = \operatorname{Cl} \cup_{X^2} C^*(x)$. Since by definition of X^2, $p^o \in \cup_{X^2} C^*(x)$, and $p^o \in X^{1*}$, we have $p^o \in \operatorname{Bnd} \hat{C}^*(p) = X^{1*}$.
If $x^o \in \operatorname{Bnd} X^1 \cap \operatorname{Bnd} X^2$, then $C(x^o) = X^1$. By definition

$C(x^o) \subset X^1$. Suppose for some $x^1 \in X^1, x^1 \notin C(x^o)$. Then $C(x^1) \supset C(x^o)$ and $x^o \in \text{Bnd } C(x^1)$, but this contradicts continuity.

For $p \in \text{Bnd } X_i^*$, $L(p)$ supports X_i. If $L(p)$ supports $C_i^*(x_i)$ for any $x_i \in X_i$, then $\hat{C}_i^*(p_i) = V_i^*$. If not, there exists some $x \in X_i$, such that $\hat{C}_i^*(p_i) = C_i^*(x_i)$.

Now we are able to prove the continuity properties.

<u>Lower hemi continuity</u>: Let B be an open set and $\hat{C}^*(p^o) \cap B \neq \emptyset$. Let $q^o \in A \cap \text{Int } C^*(p^o)$. Since $p^o \in \text{Bnd } C^*(p^o)$, $\hat{C}^*(q^o) \subset \hat{C}^*(p^o)$ and, $p^o \notin \text{Bnd } C^*(q^o)$, otherwise by definition $\hat{C}^*(p^o) \subset \hat{C}^*(q^o)$. $V^* \setminus C^*(q^o) = U$ is an open neighbourhood of p^o. For $p \in U$, we have $q^o \in \hat{C}^*(p)$, $\hat{C}^*(p) \cap B \neq \emptyset$.

<u>Closedness:</u> Let $p^s \to p^o$, $q^s \to q^o$ and $q^s \in \hat{C}^*(p^s)$, all points of $V_i^* \setminus X_i^*$. Suppose $q^o \notin \hat{C}^*(p^o)$. Hence $\hat{C}^*(p^o) \subset \hat{C}^*(q^o)$. Choose $r \in \text{Int } C^*(q^o) \setminus C^*(p^o)$. Now $\hat{C}^*(p^o) \subset \hat{C}^*(r) \subset \hat{C}^*(q^o)$ and $p^o \in \text{Int } \hat{C}^*(r)$; since $\hat{C}^*(p^o)$ cannot support more then one preference set by ass. B5, $q^o \notin \hat{C}^*(r)$.

For some $s > n$, $p^s \in \hat{C}^*(r)$ and for some $s > m$, $q^s \notin \hat{C}^*(r)$. Hence if $s > n$ and $s > m$ $q^s \notin \hat{C}^*(p^s)$, which is a contradiction.

We are now able to define the concept of <u>dual equilibrium.</u> A dual equilibrium only consists of a <u>price vector</u>, which represents a general price. Individual prices follow from this general price, using the income distribution. Commodity vectors do not explicitely occur in this definition. They can be derived from the equilibrium price.

In this definition we use the concept of <u>dual summation</u> defined in section 9 and we repeat:

$$\Sigma\, C_i^* = \bigcup_A \bigcap_I \alpha_i\, C_i^* \text{ for } A = \{\alpha_i \mid \alpha_i \geq 0 \text{ and } \Sigma\, \alpha_i = 1\}.$$

<u>Definition 16.4</u>

A <u>dual equilibrium</u> is a price vector $\bar{p}$, such that for $\bar{p}_i = \frac{1}{\lambda_i(\bar{p})}\, \bar{p}$:

$\bar{p} \in \Sigma\ \hat{C}_i^*(p_i) \cap Y^*$

$\text{Int}\ \Sigma\ \hat{C}_i^*(p_i) \cap Y^* = \emptyset.$

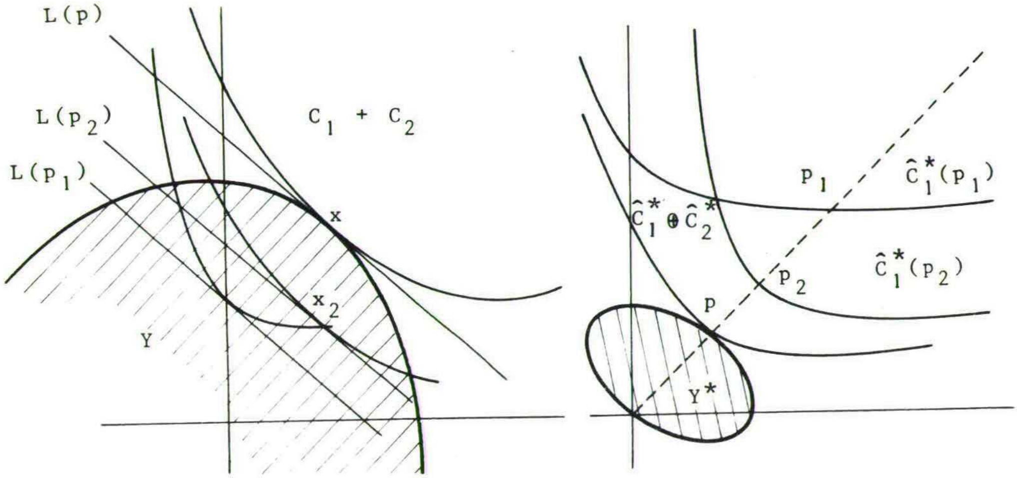


Fig. 10

Theorem 16.5

a. Given assumption b_1, b_2, b_3, b_4 and d_5:
$\bar{p}$, $\bar{x}$, $\bar{x}_i$ is an equilibrium $\Rightarrow$ $\bar{p}$ is a dual equilibrium

b. $\bar{p}$ is a dual equilibrium $\Rightarrow$ there exist $\bar{x}$ and $\bar{x}_i$, such that $\bar{p}$, $\bar{x}$, $\bar{x}_i$ is an equilibrium.

Proof.

a. By theorem 15.3, $L(\bar{p}_i)$ supports Au $C_i(x_i)$ in $\bar{x}_i$ and $L(\bar{p})$ supports Σ Au $C_i(x_i)$ and Y in $\bar{x}$. Hence $\bar{p}_i \in \text{Bnd}\ C_i^*(\bar{x}_i)$ and $\bar{p} \in \text{Bnd}\ \Sigma\ C_i^*(\bar{x}_i) \cap \text{Bnd}\ Y^*$. Now $\hat{C}_i^*(\bar{p}_i) \subset C_i^*(\bar{x}_i)$: for $\bar{p}_i \notin \text{Bnd}\ X_i^*$, this is true by definition, for $\bar{p}_i \in \text{Bnd}\ X_i^*$ this holds, because $L(\bar{p})$ does not contain any point preferred to $\bar{x}_i$. Hence $\Sigma\ \hat{C}_i^*(\bar{p}_i) \subset \Sigma\ C_i^*(x_i)$ and therefore, applying theorem 10.3, $\bar{p} \in \Sigma\ \hat{C}_i^*(\bar{p}_i) \cap Y^*$, whereas $\text{Int}\ \Sigma\ \hat{C}_i^*(\bar{p}_i) \cap Y^* = \emptyset$.

b. Since $\bar{p} \in C^* \cap Y^*$ and $\text{Int}\ C^* \cap Y^* = \emptyset$, for $C^* = \Sigma\ \hat{C}_i(\bar{p}_i)$, some hyperplane $L(\bar{x})$ separates C^* and Y^* and hence $L(\bar{p})$ separates C^{**} and Y^{**} and supports these sets in some

point $\bar{x}$. There exists $\bar{x}_i$, such that $\bar{x} = \Sigma\, \bar{x}_i$ and $L(\bar{p}_i)$ supports $\hat{C}_i^{**}(\bar{p}_i) = C^{**}(\bar{x}_i)$ in $\bar{x}_i$.
So $\bar{x}$, $\bar{x}_i$ and $\bar{p}$ are an equilibrium of the economy with preference correspondence C_i^{**} and Y^{**}, so they are also an equilibrium of the original economy by theorem 15.3.

Before we give a proof of the existence of an equilibrium for the dual economy, we first note that this dual economy can be considered independentely. This economy is defined by the concepts given in section 3. We give a set of assumptions, that follow from the assumptions given for the original economy and these assumptions are sufficient for the existence of a dual equilibrium.
In the proof of theorem 16.6 we refer to these assumptions. Theorem 16.6 ensures the existence of an equilibrium in the original economy together with theorem 16.5.

<u>Assumptions</u>. (for the dual representation of the economy)

A $\quad Y^*$ is closed and convex, $0 \in Y^*$

B1 $\quad X_i^*$ and V_i^* are closed, convex, aureoled, $0 \notin V_i^*$

B2 $\quad X_i^* \subset V_i^*$

B3 $\quad$ All sets $\hat{C}_i^*(p_i)$ are closed, convex, aureoled and $0 \in \hat{C}_i^*(p_i)$

B4 $\quad$ for $p,q \in V_i^* \setminus \text{Int}\, X_i^*$: $p_i \in \hat{C}_i^*(q_i)$ or $q_i \in \hat{C}_i^*(p_i)$ and
$q \in \hat{C}_i^*(p_i) \Rightarrow \hat{C}_i^*(q_i) \subset \hat{C}_i^*(p_i)$

B5 $\quad \hat{C}_i^*$ is closed and l.h.c.

C1 $\quad (\text{Int}\, \Sigma\, X_i^*) \cap Y^* = \emptyset$

C2 $\quad$ For all i: $(\Sigma\, X_i^* \bullet V_i^*) \cap Y^* \neq \emptyset$

C3 $\quad Y^* \cap \Sigma\, V_i^*$ is compact.

D1 $\quad$ The functions $\lambda_i(p)$ are continuous in P

D2 $\quad$ for $\mu > 0$, $\lambda_i(p) = \lambda_i(\mu p)$

D3 $\quad \Sigma\, \lambda_i(p) = 1$

D4 $\quad \frac{1}{\lambda_i(p)}\, p \notin X_i^*$ for $p \in P$

D5 $\quad \lambda_i(p) > 0$ for $p \in P$

These assumptions are implied by the ones given in section 4: A is true by definition of lower dual sets. B1 is true by definition of upper dual sets an d $V_i \neq \emptyset$ was proved in theorem 15.4. B2 and B3 hold by definition and B4 and B5 were proved in theorem 16.2. C1 follows from assumption c5, by applying theorem 10.3. C2 follows from the definition of V_i, applying theorem 10.3. C3 is implied by c3:
Since $\cup V_i$ and Y cannot be separated by some H(q,0), neither $\cup$ Au V_i and Y can be separated. So by theorem 10.2, Cl Cone $(\cap \text{ Au } V_i^*) \cap$ Coneint $Y^* \subset 0$ and therefore also Cl Cone $\Sigma V_i^* \cap$ Coneint $Y^* \subset 0$. Now the assumption follows by theorem 10.1.
The assumptions D directly follow from d.

Theorem 16.6

Given the assumptions for the dual economy there exists an equilibrium price $\bar{p}$.

Proof of theorem 16.6

By assumptions C2 and C3 the set $Y^* \cap V^*$ is non-empty and compact. Since $0 \notin V^*$, $0 \notin Y^* \cap V^*$. Any equilibrium price must be in $V^* \cap \text{Bnd } Y^* \subset V^* \cap Y^*$.[1)]
We define two functions:

$$\alpha: \text{Cl Cone } Y^* \cap V^* \setminus \{0\} \to R$$

$$p: \text{Cl Cone } Y^* \cap V^* \setminus \{0\} \to \text{Bnd } Y^* \cap V^*$$

1) Bnd Y^* is the boundary of Y^* with respect to Cone Y^*, i.e. the set $\{p \in Y^* \mid \forall \mu < 1 : \mu p \in Y^*\}$

where

$$\alpha(q) = \max\{\alpha \in R \mid \alpha\, q \in Y^*\}$$

$$p(q) = \alpha(q)q$$

Since $Y^* \cap V^*$ is convex, compact and does not contain o (by assumption A),

$$\alpha(q) > 0 \text{ and } p(q) \neq 0 \text{ if } q \in \text{Cone } Y^* \cap V^* \setminus 0$$

and both functions are continuous and α is quasi concave. Obviously $p(q) \in \text{Bnd } Y^*$, since Y^* is star shaped and V^* is aureoled, so with any arbitrary non zero price vector q of Cl Cone Y^* is associated the general price p(q) on the ray from the origin through q.
Let $H = \{p \in R^{n*} \mid p\ h = 1\}$ for $h \in R^n$, be a hyperplane which strictly separates $V^* \cap Y^*$ and $\{0\}$ and $S = H \cap \text{Cone }(V^* \cap Y^*)$ is called the set of standard prices. S is convex and compact: that it is bounded follows, by theorem 10.1 from the fact that Coneint St $H \cap$ Cl Cone $V^* \cap Y^* = \{0\}$.
We define the inverse functions

$$s : \text{Bnd } Y^* \cap V^* \to S \text{ and } \gamma = \text{Bnd } Y^* \cap V^* \to R, \text{ where}$$

$$s(q) = p^{-}(q) = \{s \mid q = p(s)\}$$

$$\gamma(q) = \frac{1}{\alpha(s(q))}$$

Now $s(q) = \gamma(q)q$ and both functions are continuous.

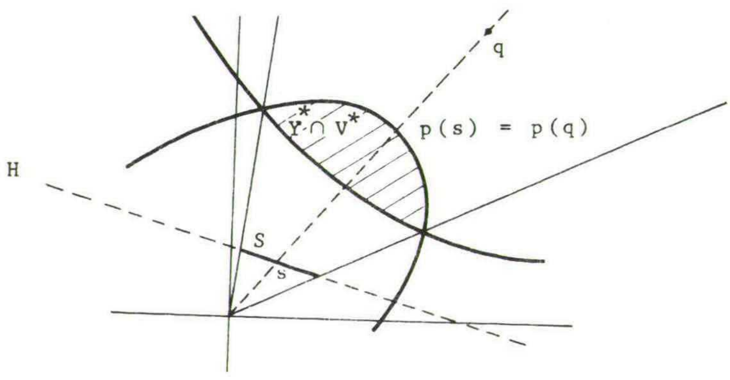


Fig. 11

An <u>individual price</u> (for the i'th individual) is related to a general price by a number μ, representing an income: $p_i = \frac{1}{\mu} p$. The income distribution function assigns for $q \in \text{Cone } Y^*$, an income $\lambda_i(q)$ to each individual. By assumption D2, $\lambda_i(q) = \lambda_i(\alpha(q)q)) = \lambda_i(p)$. With any standard price s and general price p(s) can be associated an individual price $p_i(s)$, by deflating the general price with the income. Hence we map the set of standard prices into the „individual price space". Let $p_i : S \rightarrow R^n$, where

$$p_i(s) = \frac{1}{\lambda_i(p(q))} p(s) = \frac{\alpha(s)}{\lambda_i(s)} s \text{ if } s \in S$$

The function is <u>continuous</u>. This follows from the continuity of $\alpha(s)$, p(s) and $\lambda_i(s)$ (assumption D1) and from assumption D5 which requires $\lambda_i(s) > 0$.

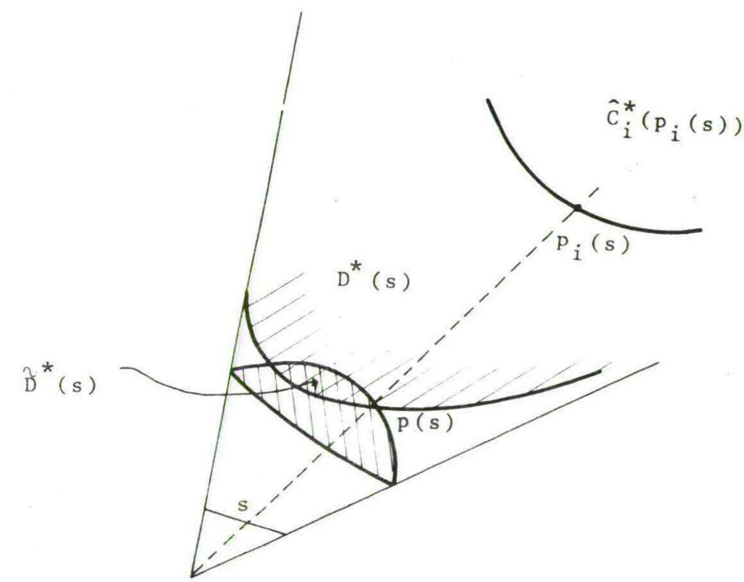


Fig. 12

We now define a correspondence $D_i^*: S \to V_i^*$

$$D_i^*(s) = \begin{cases} \hat{C}_i^*(p_i(s)) & \text{if } p_i(s) \in V_i^* \\ V_i^* & \text{if } p_i(s) \notin V_i^* \end{cases}$$

Hence to any s is assighned the set of individual prices not "better" than $p_i(s)$, or the whole set of feasible prices V_i^*. Since $\hat{C}_i^*$ is a closed, l.h.c. correspondence by B5 and $p_i(s)$ is a continuous function, its composition D_i^* is also closed, l.h.c.

Let $D^*: S \to V^*$, for $V^* = \Sigma\, V_i^*$, be the dual sum of the D_i^*

$$D^*(s) = \Sigma\, D_i^*(s)$$

By theorem 10.3 this correspondence is closed and l.h.c.

<u>Lemma a</u>.

$\forall s \in S: D^*(s) \cap Y^* \neq \emptyset$.

<u>Proof</u>.

If $\forall i: p_i(s) \in V_i^*$, then $p_i(s) \in D_i^*(s)$, $p(s) \in Y^*$ and $p(s) = \lambda_i p(s) \in \lambda_i\, D_i^*(s)$. Since $\Sigma\, \lambda_i = 1, p(s) \in \Sigma\, D_i^*(s) = \bigcup_\alpha \bigcap_I \alpha_i\, D_i^*(s)$.

If $\exists j: p_j(s) \notin V_j^*$, then $D_j^*(s) = V_j^*$. Now $D^*(s) \cap Y^* = [\sum_{j \neq i} D_i^*(s) \oplus V_j^*] \cap Y^* \supset [\sum_{j \neq i} X_i^* \oplus V_j^*] \cap Y^* \neq \emptyset$ by assumption C2.

<u>Lemma b</u>.

a. $\forall i: p_i(s) \in V_i^* \Rightarrow p(s) \in \text{Bnd}\, D^*(s)$

b. $\exists i: p_i(s) \notin V_i^* \Rightarrow p(s) \notin D^*(s)$.

Proof.

a. For all i: $p_i(s) \in \text{Bnd } \hat{C}_i^*(p_i) = \text{Bnd } D_i^*(s)$ and $p(s) = \lambda_i(s)\, p_i(s)$, or $p(s) \in \text{Bnd } \lambda_i(s)\, D_i^*(s)$. Therefore for any $\phi < 1$, $\phi p(s) \notin \lambda_i(s)\, D_i^*(s)$, for all i.

b. For j: $\frac{1}{\lambda_i(p(s))}\, p(s) \notin V_j^* = D_j^*(s)$. Suppose $p(s) \in D^*(s)$. Then for some $\mu(\mu_i > 0, \Sigma\, \mu_i = 1)$, $p(s) \in \mu\, D_i^*(p(s))$ for all i. Now $p(s) \in \mu_j\, V_j^*$, hence $\mu_j < \lambda_j$. So for some $i \neq j$ $\mu_i > \lambda_i$. But then $p(s) \notin \mu_i\, D_i^*(s)$, since $p(s) \in \text{Bnd } \lambda_i\, D_i^*(s)$.

Let $\tilde{D}^*(s) = D^*(s) \cap Y^*$. $\tilde{D}^*(s)$ is compact, convex, non-empty for all s. So it is <u>upper hemi continuous</u> and l.h.c., hence continuous.

Let

$$\beta(s) = \max\, \{\alpha(q) \mid q \in \tilde{D}^*(s)\}$$

and

$$B(s) = \{q \mid \alpha(q) = \beta(s) \text{ and } q \in \tilde{D}_i^*(s)\}$$

By the maximum theorem ([1] p. 122), we have

a. $\beta(s)$ is a continuous function

b. $B(s)$ is an u.h.c. correspondence.

We have, for all $s \in S$:

1. $\beta(s) \geq 1$

2. $B(s) \subset Y^*$

since $D_i^*(s) \cap Y^* \neq \emptyset$.

Now the points of $B(s)$ are mapped into $S, F: S \to S$ for

$$F(s) = \{r \in S \mid q \in B(s) \text{ and } q \in \text{Cone } r\}$$

$$= \{r \in S \mid q \in B(s) \text{ and } r = \gamma(q)\}$$

Since γ is a continuous function and $B(s)$ is an u.h.c. correspondence $F(s)$ is u.h.c.

Further F(s) is convex:

$$F(s) = S \cap \text{Cone } D^{*}(s) \cap \frac{1}{\beta(s)} Y^{*}$$

which is a convex set.
Now F is an u.h.c. correspondence of S into itself with convex image.
Hence we can apply Kakutani's fixed point theorem: there exists $\bar{s} \in S$, such that $\bar{s} \in F(\bar{s})$.
Now

a. It is impossible that for some i, $p_i(s) \notin V_i^*$, since in this case, by lemma 2, $p(\bar{s}) \notin D^*(\bar{s}) \supset B(\bar{s})$.

b. So by lemma b, $p(\bar{s}) \in \text{Bnd } D^*(\bar{s})$ and $\beta(\bar{s}) = \alpha(p(\bar{s})) = 1$ and $\alpha(q) \leq 1$ for all $q \in D^*(s)$. Therefore $\text{Int } D^*(\bar{s}) \cap Y^* = \emptyset$ and $p(\bar{s})$ is an equilibrium price. Q.E.D.

The argument of the proof implies that it possible to find the equilibrium by a procedure of minimizing and maximizing a continuous function.
Let $Q: S \times S \to R$, where

$$\phi(r,s) = \begin{cases} \max \{\phi \mid \frac{1}{\phi} P(r) \in D\ (s)\} & \text{if } 2\ p(r) \in D^*(s) \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

This function is continuous and, by lemma a, for each s, there exists r, such that $\phi(r,s) \geq 1$.
Also $\max_r \phi(r,s) = \beta(s)$ and $\min_s \beta(s) = 1 = \rho(\bar{s})$ for the equilibrium price $p(\bar{s})$.
Hence

<u>Corrolory</u>.

$p(\bar{s})$ is an equilibrium price for

$$\min_s \max_r \phi(r,s) = \phi(\bar{s},\bar{s}) = 1.$$

M.v.d.B.

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